Remarks on the Quantum Bohr Compactification

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Abstract

The category of locally compact quantum groups can be described as either Hopf $*$-homomorphisms between universal quantum groups, or as bicharacters on reduced quantum groups. We show how Sołtan's quantum Bohr compactification can be used to construct a "compactification" in this category. Depending on the viewpoint, different C*-algebraic compact quantum groups are produced, but the underlying Hopf $*$-algebras are always, canonically, the same. We show that a complicated range of behaviours, with C*-completions between the reduced and universal level, can occur even in the cocommutative case, thus answering a question of Sołtan. We also study such compactifications from the perspective of (almost) periodic functions. We give a definition of a periodic element in $L^\infty(\mathbb{G})$, involving the antipode, which allows one to compute the Hopf $*$-algebra of the compactification of $\mathbb{G}$; we later study when the antipode assumption can be dropped. In the cocommutative case we make a detailed study of Runde's notion of a completely almost periodic functional– with a slight strengthening, we show that for [SIN] groups this does recover the Bohr compactification of $\hat{G}$.

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1 Introduction

The Bohr, or strongly almost periodic, compactification of a topological group G is the maximal compact group G^{SAP} containing a dense homomorphic image of G. One can construct G^{SAP} by looking at the finite-dimensional unitary representation theory of G, but when G is locally compact, there is an intriguing link with Banach and C*-algebra theory. Let $AP(G)$ denote the collection of $f \in C^b(G)$ whose orbits, under the left- or right-translation actions of G on $C^b(G)$, form relatively compact subsets of $C^b(G)$ (the collection of almost periodic functions). Then $AP(G)$ is a commutative unital C*-algebra, the character space is G^{SAP}, and the group structure of G^{SAP} can be "lifted" from the group structure of G. In this picture, we never go near representation theory!

In the framework of noncommutative topology, one replaces spaces by algebras– we think of G as being represented by $C_0(G)$, and the product on G being given by a coproduct $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G)) = C^b(G \times G); \Delta(f)(s,t) = f(st)$. Then Δ is coassociative, and one can think of a "quantum semigroup" (or, more prosaically, a C*-bialgebra) as being a pair (A, Δ) where A need no longer be commutative. When A is unital, and we have the "cancellation" conditions that

$$\mathrm{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\}, \quad \mathrm{lin}\{\Delta(a)(1 \otimes b) : a, b \in A\}$$

are dense in $A \otimes A$, then we have a compact quantum group. The pioneering work of Woronowicz, [56], shows that such objects have a remarkable amount of structure, and generalise completely the theory of compact groups. Sołtan in [43] studied how to "compactify" a C*-bialgebra, and produced a very satisfactory theory, very much paralleling (and generalising) the representation–theoretic approach to constructing the classical Bohr compactification.

Going back to a locally compact group G, more abstractly, we can work with the convolution algebra $L^1(G)$, turn the dual space $L^1(G)^* = L^\infty(G)$ into an $L^1(G)$-bimodule, and look at the

functionals $F \in L^\infty(G)$ such that the orbit map $L^1(G) \to L^\infty(G); a \mapsto a \cdot F$ is a compact linear map. Then we also recover $AP(G) \subseteq C^b(G) \subseteq L^\infty(G)$. This theory has been generalised to general Banach algebras, and in particular to the Fourier algebra (firstly in [14]). However, links here with any notion of a "compactification" are very tentative.

In this paper, we have two major goals, both centred around understanding further Sołtan's construction as applied to locally compact quantum groups. These are C*-bialgebras with additional, "group-like", structure. Firstly, in a category theoretic sense, we have the inclusion functor from the category of compact groups to the category of (say) locally compact groups. The Bohr compactification is the universal arrow to this functor (see Section 2 below). Building on work of Kustermans and Ng, the recent paper [34] gave a very satisfactory picture for what morphisms between locally compact quantum groups should be. In Section 3 we show how to construct a compactification as a "universal object" in this category, see Proposition 3.4. A major technical stumbling block is that we think of a single locally compact quantum group as being represented by a number of different algebras, this paralleling the fact that for a non-amenable G, the universal group C*-algebra $C^*(G)$, and the reduced algebra $C^*_r(G)$, are different. Working at the "universal" level, the morphisms for locally compact quantum groups are just Hopf $*$-homomorphisms, but Sołtan's construction may fail to give a universal compact quantum group. Similarly, compactifying at the reduced level may give a different compact quantum group, but we show that the underlying Hopf $*$-algebras are always the same (in a canonical way), see Proposition 3.9.

In Section 4 we study our other major goal, and look at how the quantum Bohr compactification could be constructed without reference to (co)representations (thus paralleling the "almost periodic" construction of the classical Bohr compactification). For a locally compact quantum group $\mathbb{G}$, the philosophy is that the "group structure" of $\mathbb{G}$ should be enough to allow us to construct the compactification $\mathbb{G}^{\mathrm{SAP}}$ without explicitly looking at representations. We define a notion of a "periodic" element, and show how to recover this from just knowledge of the convolution algebra $L^1(\mathbb{G})$, see Proposition 4.9. We then show that for Kac algebras, or under a further hypothesis involving the antipode, this notion of periodic element allows one to construct $\mathbb{G}^{\mathrm{SAP}}$, see Section 4.3.

In Section 5 we study the Fourier algebra is further detail. In [42] Runde used Operator Space theory to define the notion of a "completely almost periodic functional". Under an injectivity hypotheses, we end up looking at the C*-algebra

$$\{x \in L^\infty(\mathbb{G}) : \Delta(x) \in L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})\},$$

where $\otimes$ here denotes the C*-algebraic spacial tensor product. In the fully quantum case, we show in Section 4.1 that the quantum $E(2)$ group gives an example to show that there is little hope of such a definition capturing the Bohr compactification. However, in Theorem 5.1 we show, in particular, that for a discrete group G this definition, when applied to the Fourier algebra $A(G)$, does recover $C^*_r(G)$ as we might hope; for the classical almost periodic definition, this was only known in the amenable case. We then study [SIN] groups, and show that a slight further strengthening of Runde's definition does allow us to recover the quantum Bohr compactification, see Theorem 5.3.

Finally, in Section 6, we study further examples. By looking again at the Fourier algebra, we answer (negatively) some conjectures of Sołtan, showing in particular that finding the quantum Bohr compactification of $C^*(G)$ and $C^*_r(G)$ may yield different completions of the same underlying Hopf $*$-algebra, and that even for the reduced $C^*_r(G)$, the resulting compact quantum group might fail to be itself reduced. This also shows that we did indeed need to be careful in Section 3! In the special cases of discrete and compact quantum groups, we show how the "extra hypotheses" which appeared in previous sections can be removed.

We start the paper in Section 2 with an introduction to the quantum groups we are interested in, and the categories they form. We finish the paper with some open problems.

1.1 Acknowledgements

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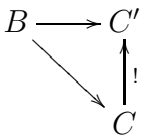
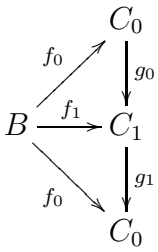
2 Categories

We take a slightly general approach to compactifications. Let $\mathfrak{B}$ be a category, and let $\mathfrak{C}$ be a full subcategory of $\mathfrak{B}$. We shall think of the objects of $\mathfrak{C}$ as being the "compact" objects of $\mathfrak{B}$ (but be aware that this has nothing to do with the, somewhat more specific, category–theoretic notion of a "compact object"). Given an object $B \in \mathfrak{B}$, a "compactification" of B is an object $C \in \mathfrak{C}$ and an arrow $B \xrightarrow{f} C$ which satisfies the following universal property: for any $C' \in \mathfrak{C}$ and any arrow $B \to C'$, then there is a *unique* arrow $C \to C'$ making the diagram commute:

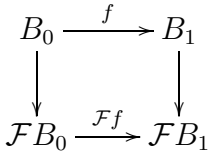


In particular, taking $C = C'$, uniqueness ensures that the identity morphism on C is the only arrow $g : C \to C$ with $gf = f$. This property ensures that compactifications, if they exist, are unique up to isomorphism. Indeed, suppose that B has two compactifications, $B \xrightarrow{f_0} C_0$ and $B \xrightarrow{f_1} C_1$. Then applying the universal property of C_0 to the arrow f_1 yields a unique $g_0 : C_0 \to C_1$ with $g_0 f_0 = f_1$. Similarly we get a unique $g_1 : C_1 \to C_0$ with $g_1 f_1 = f_0$:



Then the composition $g_1 g_0$ satisfies the relation $g_1 g_0 f_0 = f_0$, and so $g_1 g_0$ is the identity. Similarly, $g_0 g_1$ is the identity, and so C_0 and C_1 are isomorphic, as claimed.

Suppose now that every object in $\mathfrak{B}$ has a compactification; so by uniqueness, we get a map $\mathcal{F} : \mathfrak{B} \to \mathfrak{C}$. Given any arrow $B_0 \xrightarrow{f} B_1$ in $\mathfrak{B}$, we have the composition $B_0 \xrightarrow{f} B_1 \to \mathcal{F}B_1$, where $\mathcal{F}B_1 \in \mathfrak{C}$, and so the the universal property of $\mathcal{F}B_0$ gives a unique arrow $\mathcal{F}B_0 \xrightarrow{\mathcal{F}f} \mathcal{F}B_1$ making the following diagram commute



It is a simple exercise in drawing diagrams, and using uniqueness again, that $\mathcal{F}(f \circ g) = \mathcal{F}f \circ \mathcal{F}g$, that is, $\mathcal{F}$ is a functor $\mathfrak{B} \to \mathfrak{C}$.

Of course all this is well-known: our notion of a compactification is just the "universal arrow from B to the inclusion functor $\mathfrak{C} \to \mathfrak{B}$" (see [33, Chapter III]). Indeed, if compactifications exist, then we have that $\mathfrak{C}$ is "reflective" in $\mathfrak{B}$ and the compactification is simply the "reflection" (see [33,

Section IV.3]). This sort of "categorical" approach to defining the classical Bohr compactification of a group was stuided in [23, 24], and for a similar treatment of the quantum case, see the recent paper [8] (which essentially gives a treatment of Sołtan's work via abstract categorical arguments, but which does not consider the category LCQG described below).

We next introduce the two categories which shall interest us in this paper.

2.1 C*-bialgebras

Recall that a "morphism" (in the sense of Woronowicz) between C*-algebras B and A is a non-degenerate $*$-homomorphism $\theta : B \to M(A)$. Such a non-degenerate $*$-homomorphism has a unique extension to a unital, strictly continuous $*$-homomorphism $M(B) \to M(A)$, the *strict extension* of θ, which in this paper we shall tend to denote by the same symbol θ. As such, morphisms can be composed. We also tend to be slightly imprecise, and to speak of a morphism from B to A (when really the map is to $M(A)$) especially when drawing commutative diagrams.

The motivation comes from the commutative situation: if A and B are commutative, then there are locally compact Hausdorff spaces X_A, X_B with $A \cong C_0(X_A)$ and $B \cong C_0(X_B)$. Furthermore, there is a bijection between morphisms $\theta : B \to A$ and continuous maps $\phi : X_A \to X_B$ given by $\theta(f) = f \circ \phi$. If we did not consider the multiplier algebra $M(A)$, then we would have to restrict attention to proper continuous maps. See [32, 58] and perhaps especially [55, Chapter 2 exercises] for further details.

Let CSBa be the category of C*-bialgebras (A, Δ), here thought of in the general sense as A being a (not necessarily unital) C*-algebra and Δ a non-degenerate $*$-homomorphism $A \to M(A \otimes A)$ which is coassociative in the sense that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. An arrow $(A, \Delta_A) \to (B, \Delta_B)$ is then a non-degenerate $*$-homomorphism $\theta : B \to M(A)$ with $\Delta_A\theta = (\theta \otimes \theta)\Delta_B$. A non-degenerate $*$-homomorphism which intertwines the coproducts in this fashion is termed a "Hopf $*$-homomorphism" in [34]. Here we have "reversed" the arrows to generalise better from the commutative situation, as if $(A, \Delta) \in$ CSBa with A commutative, then $A = C_0(S)$ for some locally compact semigroup S with Δ induced in the usual way, $\Delta(f)(s, t) = f(st)$. Given the discussion above, morphisms restrict to the usual notion of a continuous semigroup homomorphism.

In CSBa, we shall define the "compact" objects to be the compact quantum groups in the Woronowicz sense, see the introduction and of course [56].

2.2 Locally compact quantum groups

Let LCQG be the category of locally compact quantum groups, with morphisms in the sense of [34]. Let us remark briefly on definitions. We shall follow the Kustermans and Vaes definition, see [28, 29, 30, 51].

A *locally compact quantum group* in the von Neumann algebraic setting is a Hopf-von Neumann algebra (M, Δ) equipped with left and right invariant weights. As usual, we use Δ to turn M_* into a Banach algebra, and we write the product by juxtaposition. We shall "work on the left"; so using the left invariant weight, we build the GNS space H, and a "multiplicative" unitary W acting on $H \otimes H$ (of course, the existence of a right weight is needed to show that W is unitary). There is a (in general unbounded) antipode S which admits a "polar decomposition" $S = R\tau_{-i/2}$, where R is the unitary antipode, and (τ_t) is the scaling group. There is a nonsingular positive operator P which implements (τ_t) as $\tau_t(x) = P^{it}xP^{-it}$. Then W is *manageable* with respect to P. One can develop a slightly more general theory of quantum group from such manageable (or related, "modular") multiplicative unitaries, see [45, 46, 57]. Many of the results of this paper work in this more general setting; see remarks later.

Given such a W, the space $\{(\iota \otimes \omega)W : \omega \in \mathcal{B}(H)_*\}$ is an algebra, and its closure is a C*-algebra, say A. There is a coassociative map $\Delta : A \to M(A \otimes A)$ given by $\Delta(a) = W^*(1 \otimes a)W$. If we formed W from (M, Δ) with invariant weights, then A is σ-weakly dense in M, and the two definitions of Δ agree. Similarly, $\{(\omega \otimes \iota)W : \omega \in \mathcal{B}(H)_*\}$ is norm dense in a C*-algebra $\hat{A}$, and defining $\hat{\Delta}(\hat{a}) = \hat{W}^*(1 \otimes \hat{a})\hat{W}$, we get a non-degenerate $*$-homomorphism $\hat{\Delta} : \hat{A} \to M(\hat{A} \otimes \hat{A})$, where here $\hat{W} = \Sigma W^* \Sigma$, and Σ is the tensor swap map $H \otimes H \to H \otimes H$. If we started with (M, Δ) having invariant weights, then we can construct invariant weights on $(\hat{A}'', \hat{\Delta})$. The unitary W is in the multiplier algebra $M(A \otimes \hat{A}) \subseteq \mathcal{B}(H \otimes H)$.

We write $\mathbb{G}$ for an abstract object to be thought of as a quantum group. We write $C_0(\mathbb{G}), L^\infty(\mathbb{G})$ and $L^1(\mathbb{G})$ for A, M and M_*. We also write $L^2(\mathbb{G})$ for H.

Again, the commutative examples always arise from locally compact groups with their Haar measures, and again, morphisms between C*-alegbras, which intertwine coproducts, correspond to continuous group homomorphisms. However, the cocommutative examples are of the form $C_r^*(G)$, the *reduced* group C*-algebras (and at the von Neumann algebra level, $VN(G)$) as we work with faithful weights. Then, for example, the trivial group homomorphism should correspond to trivial representation $C^*(G) \to \mathbb{C}$, but this remains bounded on $C_r^*(G)$ only for amenable G (see [38, Theorem 4.21], [25, Section 6]).

The passage from $C_r^*(G)$ to $C^*(G)$ can analogously be performed for quantum groups, see [27]. We shall write $C_0^u(\mathbb{G})$ for the *universal* C*-algebraic form of $\mathbb{G}$. A similar object can be found for manageable multiplicative unitaries, see the second part of [46].

One possible definition for a morphism in LCQG is then a non-degenerate $*$-homomorphism $C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$ which intertwines the coproduct. This was explored in [27, Section 12] where links with certain coactions of the associated L^∞ algebras was established. Previously (before the canonical definition of a locally compact quantum group had been given) Ng studied similar ideas in [35]. Unifying and extending these ideas, the paper [34] shows that Hopf $*$-homomorphisms at the universal level correspond bijectively to various other natural notions of "morphism", and we take this as a working definition of an arrow in LCQG.

In summary, an object $\mathbb{G}$ in LCQG is a locally compact quantum group, thought of as either being in reduced form $C_0(\mathbb{G})$, or in universal form $C_0^u(\mathbb{G})$. A morphism $\mathbb{G} \to \mathbb{H}$ can be described in a number of equivalent ways:

- As a non-degenerate $*$-homomorphism $C_0^u(\mathbb{H}) \to C_0^u(\mathbb{G})$ which intertwines the coproduct (that is, a Hopf $*$-homomorphism between universal quantum groups).
- As a *bicharacter* which is a unitary $U \in M(C_0(\mathbb{G}) \otimes C_0(\hat{\mathbb{H}}))$ which satisfies $(\Delta_{\mathbb{G}} \otimes \iota)(U) = U_{13}U_{23}$ and $(\iota \otimes \Delta_{\hat{\mathbb{H}}})(U) = U_{13}U_{12}$.

As above, we "reverse" the arrows from [34], so as to better generalise from the commutative situation. Furthermore, for bicharacters, we have translated "to the left", as we are working with left Haar weights, and thus left multiplicative unitaries (which form the identity morphisms in this category).

In LCQG we shall define the "compact objects" to be those $\mathbb{G}$ with $C_0(\mathbb{G})$ (or, equivalently, $C_0^u(\mathbb{G})$) being unital. Those $C_0(\mathbb{G})$ thus arising are precisely the reduced compact quantum groups, [5, Section 2]. Let CQG be the full subcategory of compact quantum groups. Henceforth, we shall write $C(\mathbb{G})$ and $C^u(\mathbb{G})$ to stress that the algebra is unital.

To finish, notice that the relation between LCQG and CSBa is slightly involved. If we take the concrete realisation of LCQG as having objects of the form $C_0^u(\mathbb{G})$ and morphisms described by Hopf $*$-homomorphisms, then LCQG becomes a full subcategory of CSBa, and the compact objects agree. However, this viewpoint is slightly misleading, as for example $C_0(\mathbb{G})$ will be an object in CSBa different from $C_0^u(\mathbb{G})$, whereas we would generally regard these as being "the same" quantum

group. Furthermore, of course, CSBa contains a great many objects which don't arise from LCQG in any fashion. Of interest from the viewpoint of Section 2 is that the "compact" objects do correspond, all be it in a many-to-one fashion.

2.3 Sołtan's Bohr compactification

In [43], Sołtan showed that in CSBa, compactifications always exist. We shall shortly give a full account of his theory, but for now let us make some brief comments. Given an object $\mathcal{G} = (A, \Delta_A)$ in CSBa, we construct a certain unital C*-subalgebra $\mathbb{AP}(\mathcal{G})$ in $M(A)$ such that (the strict extension of) Δ_A restricts to a coproduct $\Delta_{\mathbb{AP}(\mathcal{G})}$ on $\mathbb{AP}(\mathcal{G})$. Then $\mathfrak{b}\mathcal{G} = (\mathbb{AP}(\mathcal{G}), \Delta_{\mathbb{AP}(\mathcal{G})})$ is the compactification of $\mathcal{G}$ (termed the *quantum Bohr compactification* of $\mathcal{G}$ in [43, Definition 2.14]).

Given the concrete realisation of LCQG as a full subcategory of CSBa, given $\mathbb{G}$ we can form $\mathbb{AP}(\mathbb{G})$ by applying Sołtan's theory to $C_0^u(\mathbb{G})$. However, the obvious problem is that $(\mathbb{AP}(\mathbb{G}), \Delta_{\mathbb{AP}(\mathbb{G})})$, while a compact quantum group, might fail to be a *universal* quantum group, and hence would not be a member of LCQG (viewed as a subcategory of CSBa). Indeed, this can even occur when $\mathbb{G}$ is cocommutative, see Section 6.2 below.

In the next section, we instead show how to adapt Sołtan's ideas to construct a compactification in LCQG. We will also show that while the resulting C*-algebra picture is slightly complicated, the underlying (unique) dense Hopf $*$-algebra can be constructed in a number of equivalent ways, starting from either $C_0^u(\mathbb{G})$ or from $C_0(\mathbb{G})$.

3 Compactification in LCQG

In this section, we shall show how to construct compactifications in LCQG, by somewhat directly applying Sołtan's construction. Thus we first summarise Sołtan's work in [43].

Let $\mathcal{G} = (A, \Delta)$ be an object of CSBa. A (finite-dimensional) *bounded representation* of $\mathcal{G}$ is an element T of $M(A) \otimes \mathcal{B}(H)$, where H is a finite-dimensional Hilbert space, with $(\Delta \otimes \iota)T = T_{12}T_{13}$, and such that T is invertible. Equivalently, we could term such an object a *invertible corepresentation* of A (with the Δ being clear from context) and we shall mostly stick to this latter convention (to avoid confusing $C_0(\mathbb{G})$ with $C_0^u(\mathbb{G})$, and because we want to stress the "invertible" aspect). There are obvious notions of taking the direct sum, and tensor product, of invertible corepresentations.

If we take a basis of $\mathcal{B}(H)$, then we establish an isomorphism $\mathcal{B}(H) \cong \mathbb{M}_n$, and thus identify T with $(T_{ij}) \in \mathbb{M}_n(M(A))$. Then T needs to be invertible, and to satisfy

$$\Delta(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj} \quad \text{for all } i, j.$$

Let $T^\top = (T_{ji})$ be the "transpose" of T; this is always an "anti-corepresentation". We shall say that T is *admissible* if $T^\top$ is invertible.

Remark 3.1. If A is commutative, then T is invertible if and only if $T^\top$ is, but in general the admissible corepresentations form a strict subset of the collection of invertible corepresentations. In CSBa there are counter-examples ([43, Remark 2.10] which references [54, Section 4]) but we do not know the answer for $C_0(\mathbb{G})$ (and/or $C_0^u(\mathbb{G})$) for $\mathbb{G} \in$ LCQG; see Conjecture 7.2 at the end.

The linear span of the elements T_{ij} form the collection of *matrix elements* of T. By taking direct sums and tensor products, and using the trivial representation, one can show that the collection of matrix elements of invertible corepresentations forms a unital subalgebra of $M(A)$, see [43, Proposition 2.5].

Harder to show (as we now use Woronowicz's work in [61]) is that when T is an admissible corepresentation, and B_T denotes the C*-algebra generated by the matrix elements of T, then Δ restricts to a map $B_T \to B_T \otimes B_T$, and $(B_T, \Delta|_{B_T})$ is a compact quantum group, see [43, Proposition 2.7]. It follows, see [43, Corollary 2.9], that T is similar to a unitary corepresentation (again, in CSBa the converse is not true, see [43, Remark 2.10]). One can now show that admissible corepresentations are stable under tensor product, and it follows that the set of all matrix elements of admissible corepresentations of A, say $\mathcal{AP}(A)$, forms a unital $*$-subalgebra of $M(A)$, see [43, Proposition 2.12].

Remark 3.2. Let T be a bounded corepresentation which is similar to a unitary representation U. Then it is elementary to see that $T^\top$ is invertible if and only if $U^\top$ is invertible. It follows that if we are interested in the matrix elements of admissible corepresentations, it is no loss of generality to study only admissible, *unitary* corepresentations.

We shall again abuse notation slightly (the map Δ being implicit) and write $\mathbb{AP}(A)$ for the closure of $\mathcal{AP}(A)$. Thus $\mathbb{AP}(A)$ is a unital C*-algebra and Δ restricts to $\mathbb{AP}(A)$ to give a compact quantum group $\mathfrak{b}A = (\mathbb{AP}(A), \Delta_{\mathbb{AP}(A)})$. By [5, Appendix A] we know that a compact quantum group admits a unique dense Hopf $*$-algebra. By combining this with [43, Corollary 2.15] we see that for $\mathbb{AP}(A)$, this Hopf $*$-algebra is simply $\mathcal{AP}(A)$. Finally, [43, Theorem 3.1] shows that $\mathbb{AP}(A)$ satisfies the correct universal property to be, in the sense of Section 2, the compactification of (A, Δ) in CSBa.

Remark 3.3. Let (A, Δ_A) and (B, Δ_B) be C*-bialgebras and $\theta : A \to M(B)$ a Hopf $*$-homomorphism. If $U \in M(A) \otimes \mathbb{M}_n$ is an admissible corepresentation, then $V = (\theta \otimes \iota)(U)$ will also be admissible (as, for example, $\overline{V} = (\theta \otimes \iota)(\overline{U})$ will have inverse $(\theta \otimes \iota)(\overline{U}^{-1})$). It follows that $\theta(\mathcal{AP}(A)) \subseteq \mathcal{AP}(B)$, and it is from this observation that we see that $\mathbb{AP}(A)$ has the correct universal property.

3.1 For locally compact quantum groups

Let $\mathbb{G}$ be an object of LCQG. Apply Sołtan's theory to the universal quantum group $C_0^u(\mathbb{G})$, to yield a compact quantum group $\mathbb{AP}(C_0^u(\mathbb{G}))$. This defines an object $\mathbb{K}$ of LCQG which is compact. Indeed, $C(\mathbb{K})$ is the *reduced version* of $\mathbb{AP}(C_0^u(\mathbb{G}))$, see [5, Theorem 2.1]. This can be formed as the quotient of $\mathbb{AP}(C_0^u(\mathbb{G}))$ by the null-ideal of the Haar state. Alternatively, we can start with the Hopf $*$-algebra $\mathcal{AP}(C_0^u(\mathbb{G}))$, which also carries a Haar state. Using this we can form the Hilbert space $L^2(\mathbb{K})$, and then we can identify $\mathcal{AP}(C_0^u(\mathbb{G}))$ with a $*$-algebra of operators on $L^2(\mathbb{K})$. Then $C(\mathbb{K})$ is the closure of $\mathcal{AP}(C_0^u(\mathbb{G}))$ in $\mathcal{B}(L^2(\mathbb{K}))$. See [5, Theorem 2.11].

We can now form $C^u(\mathbb{K})$, the universal version of $\mathbb{K}$, by following [27]. Alternatively, one can follow [5, Section 3]; a little work shows that these are equivalent constructions. In particular, we can think of $C^u(\mathbb{K})$ as being the universal enveloping C*-algebra of $\mathcal{AP}(C_0^u(\mathbb{G}))$, and so there is a surjective $*$-homomorphism $\Lambda^u_{\mathbb{AP}} : C^u(\mathbb{K}) \to \mathbb{AP}(C_0^u(\mathbb{G}))$ which intertwines the coproducts. The composition

$$C^u(\mathbb{K}) \xrightarrow{\Lambda^u_{\mathbb{AP}}} \mathbb{AP}(C_0^u(\mathbb{G})) \hookrightarrow M(C_0^u(\mathbb{G}))$$

is hence a Hopf $*$-homomorphism, and so defines an arrow $\mathbb{G} \to \mathbb{K}$ in LCQG. As we might hope, we have the following result:

Proposition 3.4. *With the arrow $\mathbb{G} \to \mathbb{K}$ just defined, $\mathbb{K}$ is the compactification of $\mathbb{G}$ in LCQG.*

Before we can prove this result, we need to study in detail the ideas of [34] and [27], as applied to compact quantum groups.

3.2 Morphisms and lifts

Let us recall some notions related to one and two-sided "universal bicharacters" (to use the language of [34]). We shall follow [27], but analogous results are shown in [34, 46]. Let $\mathbb{G}$ be a compact quantum group and form the "universal" algebra $C_0^u(\mathbb{G})$ together with the reducing morphism $\Lambda_{\mathbb{G}} : C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ (which is denoted by π in [27]). There is a unitary $\mathcal{V}_{\mathbb{G}} = \mathcal{V} \in M(C_0^u(\mathbb{G}) \otimes C_0(\hat{\mathbb{G}}))$ such that $(\Delta_u \otimes \iota)(\mathcal{V}) = \mathcal{V}_{13}\mathcal{V}_{23}$ and $(\iota \otimes \hat{\Delta})(\mathcal{V}) = \mathcal{V}_{13}\mathcal{V}_{12}$; we think of $\mathcal{V}$ as being a variant of W with its left-leg in $C_0^u(\mathbb{G})$; indeed, $(\Lambda_{\mathbb{G}} \otimes \iota)(\mathcal{V}) = W$. Similarly, there is $\mathcal{U} \in M(C_0^u(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{G}}))$, a fully universal version of W.

We now recall some results from [34]. Let (A, Δ) be a C*-bialgebra, and let $\mathbb{G}$ be a locally compact quantum group. We shall say that $U \in M(A \otimes C_0(\hat{\mathbb{G}}))$ is a *bicharacter* if U is unitary, $(\Delta \otimes \iota)(U) = U_{13}U_{23}$ and $(\iota \otimes \Delta_{\hat{\mathbb{G}}})(U) = U_{13}U_{12}$. Then [34, Proposition 4.2] shows that there is a bijection between such bicharacters U, and Hopf $*$-homomorphisms $\phi : C_0^u(\mathbb{G}) \to A$, the link being that $U = (\phi \otimes \iota)(\mathcal{V}_{\mathbb{G}})$.

For locally compact quantum groups $\mathbb{G}, \mathbb{H}$, we can similarly define the notion of a bicharacter in $M(C_0^u(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{H}}))$. Then [34, Proposition 4.7] shows that for any bicharacter $U \in M(C_0(\mathbb{G}) \otimes C_0(\hat{\mathbb{H}}))$, there is a unique bicharacter $V \in M(C_0^u(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{H}}))$ with $(\Lambda_{\mathbb{G}} \otimes \Lambda_{\hat{\mathbb{H}}})(V) = U$. We call V the "lift" of U; hence $\mathcal{U}$ is the lift of W.

The following is only implicit in [34] (having been rather more explicit in an early preprint[1] of that paper) so we give the short argument.

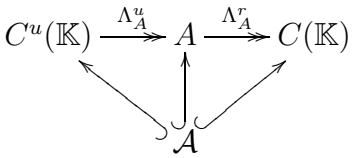
Proposition 3.5. *Given locally compact quantum groups $\mathbb{G}, \mathbb{H}$ and a Hopf $*$-homomorphism $\theta : C_0^u(\mathbb{G}) \to C_0(\mathbb{H})$, there is a unique Hopf $*$-homomorphism $\theta_0 : C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$ with $\theta = \Lambda_{\mathbb{H}}\theta_0$.*

Proof. Let $U \in M(C_0(\mathbb{H}) \otimes C_0(\hat{\mathbb{G}}))$ be the unique bicharacter associated with θ, and let $V \in M(C_0^u(\mathbb{H}) \otimes C_0^u(\hat{\mathbb{G}}))$ be the unique "lift". Then $(\iota \otimes \Lambda_{\hat{\mathbb{G}}})(V)$ is a bicharacter, and so gives a unique $\theta_0 : C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$. Then observe that $(\iota \otimes \Lambda_{\hat{\mathbb{G}}})(V) = (\theta_0 \otimes \iota)(\mathcal{V}_{\mathbb{G}})$. It follows that

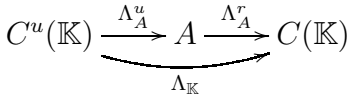
$$(\Lambda_{\mathbb{H}}\theta_0 \otimes \iota)(\mathcal{V}_{\mathbb{G}}) = (\Lambda_{\mathbb{H}} \otimes \Lambda_{\hat{\mathbb{G}}})(V) = U = (\theta \otimes \iota)(\mathcal{V}_{\mathbb{G}}).$$

As $\{(\iota \otimes \omega)(\mathcal{V}_{\mathbb{G}}) : \omega \in L^1(\hat{\mathbb{G}})\}$ is dense in $C_0^u(\mathbb{G})$, it follows that $\Lambda_{\mathbb{H}}\theta_0 = \theta$. Uniqueness follows in a similar way to [34, Lemma 50], [27, Lemma 6.1], compare Lemma 3.6 below. □

Now let A be a compact quantum group, not assumed to be reduced or universal. Let $\mathbb{K}$ be the abstract compact quantum group determined by A, so that $C(\mathbb{K})$ is the reduced version of A, and $C^u(\mathbb{K})$ is the universal version of A. As above, if $\mathcal{A}$ is the unique dense Hopf $*$-algebra of A, then $C(\mathbb{K})$ is the completion of $\mathcal{A}$ determined by the Haar state, and $C^u(\mathbb{K})$ is the universal C*-algebra completion of $\mathcal{A}$. Let $\Lambda_A^u : C^u(\mathbb{K}) \to A$ and $\Lambda_A^r : A \to C(\mathbb{K})$ be the surjective Hopf $*$-homomorphisms which make the following diagram commute:



We also have the surjective Hopf $*$-homomorphism $\Lambda_{\mathbb{K}} : C^u(\mathbb{K}) \to C(\mathbb{K})$. As this also respects the inclusion of $\mathcal{A}$ into $C_0^u(\mathbb{K})$ and $C_0(\mathbb{K})$, it follows that we have a further commutative diagram:



[1] See http://arxiv.org/abs/1011.4284v1

The following lemma uses similar techniques to [27, Result 6.1].

Lemma 3.6. *Let (A, Δ_A) and Λ_A^r be as above. Let $\mathbb{H}$ be a locally compact quantum group. If $\pi_1, \pi_2 : C_0^u(\mathbb{H}) \to A$ are Hopf $*$-homomorphisms such that $\Lambda_A^r \pi_1 = \Lambda_A^r \pi_2$, then $\pi_1 = \pi_2$.*

Proof. Consider the universal left regular representation $\mathcal{V}$ for $\mathbb{K}$. By [27, Proposition 6.2] we have that

$$(\iota \otimes \Lambda_{\mathbb{K}})\Delta_{\mathbb{K}}^u(x) = \mathcal{V}^*(1 \otimes \Lambda_{\mathbb{K}}(x))\mathcal{V} \qquad (x \in C^u(\mathbb{K})).$$

Set $U = (\Lambda_A^u \otimes \iota)(\mathcal{V}) \in M(A \otimes C_0(\hat{\mathbb{K}}))$, so that

$$U^*(1 \otimes \Lambda_{\mathbb{K}}(x))U = (\Lambda_A^u \otimes \Lambda_{\mathbb{K}})\Delta_{\mathbb{K}}^u(x) = (\iota \otimes \Lambda_A^r)\Delta_A(\Lambda_A^u(x)) \qquad (x \in C^u(\mathbb{K})).$$

It follows that

$$U^*(1 \otimes \Lambda_A^r(x))U = (\iota \otimes \Lambda_A^r)\Delta_A(x) \qquad (x \in A).$$

We remark that we could construct U purely using compact quantum group techniques, compare equation (5.10) in [56] (and remember that we work with left multiplicative unitaries).

Now let $\pi : C_0^u(\mathbb{H}) \to A$ be a Hopf $*$-homomorphism. Let $\mathcal{U}$ be the universal bicharacter of $\mathbb{H}$ and set $V = (\pi \otimes \iota)(\mathcal{U}) \in M(A \otimes C_0^u(\hat{\mathbb{H}}))$. Then

$$(\Delta_A \otimes \iota)(V) = (\pi \otimes \pi \otimes \iota)(\Delta_{\mathbb{H}}^u \otimes \iota)(\mathcal{U}) = V_{13}V_{23}.$$

By combining the previous two displayed equations, we see that

$$V_{13}((\Lambda_A^r \otimes \iota)(V))_{23} = ((\iota \otimes \Lambda_A^r)\Delta_A \otimes \iota)(V) = U_{12}^*((\Lambda_A^r \otimes \iota)(V))_{23}U_{12},$$

and so

$$V_{13} = U_{12}^*((\Lambda_A^r \otimes \iota)(V))_{23}U_{12}((\Lambda_A^r \otimes \iota)(V))_{23}^*.$$

It follows that V is determined by $(\Lambda_A^r \otimes \iota)(V) = (\Lambda_A^r \pi \otimes \iota)(\mathcal{U})$, that is, V is determined by $\Lambda_A^r \pi$. By [27, Corollary 6.1], $(\iota \otimes \Lambda_{\hat{\mathbb{H}}})(\mathcal{U}) = \mathcal{V}_{\mathbb{H}}$, and by the remarks after [27, Proposition 5.1], we know that $\{(\iota \otimes \omega)(\mathcal{V}_{\mathbb{H}}) : \omega \in L^1(\hat{\mathbb{H}})\}$ is dense in $C_0^u(\mathbb{H})$. Thus π is determined by knowing

$$\pi((\iota \otimes \omega)(\mathcal{V}_{\mathbb{H}})) = \pi((\iota \otimes \omega\Lambda_{\hat{\mathbb{H}}})(\mathcal{U})) = (\iota \otimes \omega\Lambda_{\hat{\mathbb{H}}})(V).$$

We conclude that π is determined uniquely by knowing V, and in turn V is uniquely determined by knowing $\Lambda_A^r \pi$. The result follows. □

3.3 Back to compactifications

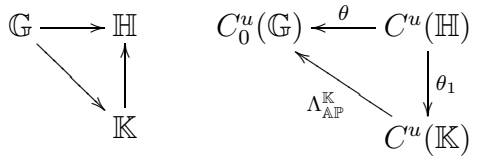
We are now in a position to prove Proposition 3.4. Let $\mathbb{K}$ be the compact quantum defined by $\mathbb{AP}(C_0^u(\mathbb{G}))$. Let $\Lambda_{\mathbb{AP}}^{\mathbb{K}} : C^u(\mathbb{K}) \to M(C_0^u(\mathbb{G}))$ be the Hopf $*$-homomorphism defining the arrow $\mathbb{G} \to \mathbb{K}$, and let $\Lambda_{\mathbb{AP}}^r : \mathbb{AP}(C_0^u(\mathbb{G})) \to C(K)$ be the "reducing morphism" considered in the previous section.

Proof of Proposition 3.4. Let $\mathbb{H}$ be compact in LCQG, and let $\mathbb{G} \to \mathbb{H}$ be an arrow. We have to show that this factors through $\mathbb{G} \to \mathbb{K}$. Let the arrow $\mathbb{G} \to \mathbb{H}$ correspond to the Hopf $*$-homomorphism $\theta : C^u(\mathbb{H}) \to C_0^u(\mathbb{G})$. As $\mathbb{AP}(C_0^u(\mathbb{G}))$ is a compactification, compare Remark 3.3, it follows that $\theta(C^u(\mathbb{H})) \subseteq \mathbb{AP}(C_0^u(\mathbb{G})) \subseteq M(C_0^u(\mathbb{G}))$. So the composition

$$C^u(\mathbb{H}) \xrightarrow{\theta} \mathbb{AP}(C_0^u(\mathbb{G})) \xrightarrow{\Lambda_{\mathbb{AP}}^r} C(\mathbb{K})$$

makes sense, and is a Hopf $*$-homomorphism $\theta_0 : C^u(\mathbb{H}) \to C(\mathbb{K})$. Let $\theta_1 : C^u(\mathbb{H}) \to C^u(\mathbb{K})$ be the unique lift given by Proposition 3.5. This defines an arrow $\mathbb{K} \to \mathbb{H}$ in LCQG.

We now claim that the following diagrams commute:



By the definition of arrows in LCQG, one diagram commutes if and only if the other does. However, we calculate that

$$\Lambda^r_{\mathbb{AP}}\theta = \theta_0 = \Lambda_{\mathbb{K}}\theta_1 = \Lambda^r_{\mathbb{AP}}\Lambda^{\mathbb{K}}_{\mathbb{AP}}\theta_1.$$

By Lemma 3.6, it follows that $\theta = \Lambda^{\mathbb{K}}_{\mathbb{AP}}\theta_1$, as required. □

Definition 3.7. Given $\mathbb{G}$, let the resulting compact quantum group $\mathbb{K}$ be denoted by $\mathbb{G}^{\mathrm{SAP}}$, the *strongly almost periodic* compactification of $\mathbb{G}$.

One could equally well call this the "Bohr compactification" of $\mathbb{G}$, but this terminology would clash with that used by Sołtan in [43] (because $\mathbb{G}^{\mathrm{SAP}}$ is an abstract quantum group, not in general a concrete sub-C*-bialgebra of $M(C_0(\mathbb{G}))$). Our terminology is inspired by that for semigroups, see [7, Section 4.3] (for "reasonable" semigroups, the "strongly almost periodic compactification" is the universal compact group compactification, while the "almost periodic compactification" is the universal compact *semigroup* compactification. For topological groups, the notions coincide: it would be interesting to investigate analogous ideas for C*-bialgebras).

Remark 3.8. Recall from Section 2 that for an arrow $\mathbb{G} \to \mathbb{H}$ in LCQG we have a unique arrow $\mathbb{G}^{\mathrm{SAP}} \to \mathbb{H}^{\mathrm{SAP}}$. If $\mathbb{G} \to \mathbb{H}$ is given by a Hopf $*$-homomorphism $\theta : C_0^u(\mathbb{H}) \to C_0^u(\mathbb{G})$, and $\mathbb{G}^{\mathrm{SAP}} \to \mathbb{H}^{\mathrm{SAP}}$ is given by $\theta^{\mathrm{SAP}} : C^u(\mathbb{H}^{\mathrm{SAP}}) \to C^u(\mathbb{G}^{\mathrm{SAP}})$, then by construction, this is the unique Hopf $*$-homomorphism making the following diagram commute:

$$\begin{array}{ccccc} C^u(\mathbb{H}^{\mathrm{SAP}}) & \xrightarrow{\Lambda^u_{\mathbb{AP}}} & \mathbb{AP}(C_0^u(\mathbb{H})) & \hookrightarrow & M(C_0^u(\mathbb{H})) \\ \downarrow{\scriptstyle \theta^{\mathrm{SAP}}} & & & & \downarrow{\scriptstyle \theta} \\ C^u(\mathbb{G}^{\mathrm{SAP}}) & \xrightarrow{\Lambda^u_{\mathbb{AP}}} & \mathbb{AP}(C_0^u(\mathbb{G})) & \hookrightarrow & M(C_0^u(\mathbb{G})) \end{array}$$

Now, the composition $\mathbb{AP}(C_0^u(\mathbb{H})) \to M(C_0^u(\mathbb{H})) \xrightarrow{\theta} M(C_0^u(\mathbb{G}))$ is a Hopf $*$-homomorphism, and so, again by [43, Theorem 3.1], it follows that the image is a subset of $\mathbb{AP}(C_0^u(\mathbb{G}))$. So actually θ^{SAP} drops to a Hopf $*$-homomorphism $\mathbb{AP}(C_0^u(\mathbb{H})) \to \mathbb{AP}(C_0^u(\mathbb{G}))$; that is, provides an arrow in the middle vertical in the diagram above.

Indeed, by Remark 3.3 we know that $\theta(\mathcal{AP}(C_0^u(\mathbb{H}))) \subseteq \mathcal{AP}(C_0^u(\mathbb{G}))$, and so θ restricts to give a map $\mathbb{AP}(C_0^u(\mathbb{H})) \to \mathbb{AP}(C_0^u(\mathbb{G}))$; this is the map considered in the previous paragraph. By composing with the inclusion $\mathcal{AP}(C_0^u(\mathbb{G})) \to C^u(\mathbb{G}^{\mathrm{SAP}})$ we obtain a Hopf $*$-homomorphism $\mathcal{AP}(C_0^u(\mathbb{H})) \to C^u(\mathbb{G}^{\mathrm{SAP}})$. As $C^u(\mathbb{H}^{\mathrm{SAP}})$ is the universal C*-algebra generated by $\mathcal{AP}(C_0^u(\mathbb{H}))$, we hence obtain a map $C^u(\mathbb{H}^{\mathrm{SAP}}) \to C^u(\mathbb{G}^{\mathrm{SAP}})$, and by tracing the construction in the proof of Proposition 3.4, we find that this map is indeed θ^{SAP}.

We next investigate what would happen if we used $\mathbb{AP}(C_0(\mathbb{G}))$ instead of $\mathbb{AP}(C_0^u(\mathbb{G}))$.

Proposition 3.9. *Let $\mathbb{G}$ be a locally compact quantum group. Consider the Hopf $*$-algebras $\mathcal{AP}(C_0(\mathbb{G}))$ and $\mathcal{AP}(C_0^u(\mathbb{G}))$, which we can consider as subalgebras of $M(C_0^u(\mathbb{G}))$ and $M(C_0(\mathbb{G}))$, respectively. Then the strict extension of $\Lambda : C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ restricts to form a bijection $\mathcal{AP}(C_0^u(\mathbb{G})) \to \mathcal{AP}(C_0(\mathbb{G}))$. In particular, $C(\mathbb{G}^{\mathrm{SAP}})$ is also the reduced version of the compact quantum group $\mathbb{AP}(C_0(\mathbb{G}))$.*

Proof. By construction (see after Definition 3.2) $\mathcal{AP}(C_0(\mathbb{G}))$ is merely the set of elements of admissible corepresentations of $C_0(\mathbb{G})$; and similarly for $\mathcal{AP}(C_0^u(\mathbb{G}))$. As Λ is a Hopf-$*$-homomorphism, it is clear that $\Lambda(\mathcal{AP}(C_0^u(\mathbb{G}))) \subseteq \mathcal{AP}(C_0(\mathbb{G}))$.

Conversely, and with reference to Remark 3.2, let U_0 be an admissible unitary corepresentation of $C_0(\mathbb{G})$, and let U be the unique lift to a unitary corepresentation of $C_0^u(\mathbb{G})$. Our aim is to show that U is admissible, from which it will follow that $\Lambda(\mathcal{AP}(C_0^u(\mathbb{G}))) = \mathcal{AP}(C_0(\mathbb{G}))$.

It is easy to see that U_0^T is invertible if and only if $\overline{U_0} = (U_{0,ij}^*)_{i,j=1}^n$ is invertible. Now, $\overline{U_0}$ is a corepresentation (and not an anti-corepresentation), and from the theory of compact quantum (matrix) groups we know that $\overline{U_0}$ is similar to a unitary, so there is a scalar matrix F with $V_0 = F^{-1}\overline{U_0}F$ unitary. Let V be the unique lift to $C_0^u(\mathbb{G})$. We shall show that $\overline{U} = FVF^{-1}$, which is invertible, showing that U is admissible as claimed.

We now argue as in the proof of Lemma 3.6. For i, j, we have that

$$\begin{aligned}\sum_k U_{ik} \otimes U_{0,kj} &= (\iota \otimes \Lambda)\Delta_u(U_{ij}) = \mathcal{V}^*(1 \otimes U_{0,ij})\mathcal{V} = \mathcal{V}^*(1 \otimes (FV_0F^{-1})_{ij}^*)\mathcal{V} \\ &= \sum_{s,t} \overline{F_{is}}\overline{F_{tj}^{-1}}\mathcal{V}^*(1 \otimes V_{0,st}^*)\mathcal{V} = \sum_{s,t} \overline{F_{is}}\overline{F_{tj}^{-1}}(\iota \otimes \Lambda)\Delta_u(V_{st}^*) \\ &= \sum_{s,t,k} \overline{F_{is}}\overline{F_{tj}^{-1}}V_{sk}^* \otimes V_{0,kt}^* = \sum_k (FV)_{ik}^* \otimes (V_0F^{-1})_{kj}^* \\ &= \sum_k (FV)_{ik}^* \otimes (F^{-1}\overline{U_0})_{kj}^* = \sum_k (FV)_{ik}^* \otimes (\overline{F^{-1}}U_0)_{kj}.\end{aligned}$$

It then follows that as U_0 is unitary, for each i, r,

$$\begin{aligned}\sum_{k,j} U_{ik} \otimes U_{0,kj}U_{0,rj}^* &= \sum_k U_{ik} \otimes \delta_{k,r}1 = U_{ir} \otimes 1 \\ &= \sum_{k,j} (FV)_{ik}^* \otimes (\overline{F^{-1}}U_0)_{kj}(U_0^*)_{jr} = \sum_k (FV)_{ik}^* \otimes \overline{F^{-1}}_{kr} = (FVF^{-1})_{ir}^* \otimes 1.\end{aligned}$$

Hence $\overline{U} = FVF^{-1}$ as claimed.

So we have shown that admissible unitary corepresentations of $C_0(\mathbb{G})$ lift to admissible unitary corepresentations of $C_0^u(\mathbb{G})$. Finally, we argue as in [12, Section 1.2]. Let $\mathcal{B}_u \subseteq M(C_0^u(\mathbb{G}))$ denote the space of elements of all unitary corepresentations of $C_0^u(\mathbb{G})$. By the universal property of $\mathcal{U}$, it follows that

$$\mathcal{B}_u = \{(\iota \otimes \omega \circ \phi)(\mathcal{U}) : \phi : C_0^u(\hat{\mathbb{G}}) \to \mathcal{B}(H) \text{ is a non-degenerate } *\text{-homomorphism }, \omega \in \mathcal{B}(H)_*\}.$$

Similarly define $\mathcal{B} \subseteq M(C_0(\mathbb{G}))$, so

$$\mathcal{B} = \{(\iota \otimes \omega \circ \phi)(\hat{\mathcal{V}}) : \phi : C_0^u(\hat{\mathbb{G}}) \to \mathcal{B}(H) \text{ is a non-degenerate } *\text{-homomorphism }, \omega \in \mathcal{B}(H)_*\}.$$

Then Λ restricts to a surjection $\mathcal{B}_u \to \mathcal{B}$, because $(\Lambda \otimes \iota)(\mathcal{U}) = \hat{\mathcal{V}}$. We claim that $\Lambda : \mathcal{B}_u \to \mathcal{B}$ is an injection, from which it will follow certainly that $\Lambda : \mathcal{AP}(C_0^u(\mathbb{G})) \to \mathcal{AP}(C_0(\mathbb{G}))$ is injective, as required.

Let $a = (\iota \otimes \omega \circ \phi)(\mathcal{U}) \in \mathcal{B}_u$ be non-zero. So $\mu = \omega \circ \phi \in C_0^u(\hat{\mathbb{G}})^*$ is non-zero. Now, $C_0^u(\hat{\mathbb{G}})$ is the closed linear span of $\{(\tau \otimes \iota)(\hat{\mathcal{V}}) : \tau \in L^1(\mathbb{G})\}$, and as $\mu \neq 0$, there is $\tau \in L^1(\mathbb{G})$ with $\langle \mu, (\tau \otimes \iota)(\hat{\mathcal{V}})\rangle \neq 0$. Thus $\Lambda(a) = (\iota \otimes \mu)(\hat{\mathcal{V}}) \neq 0$, as required.

As $C(\mathbb{G}^{\mathrm{SAP}})$ is the completion of $\mathcal{AP}(C_0^u(\mathbb{G}))$ for the norm coming from the action on $L^2(\mathbb{G}^{\mathrm{SAP}})$, it follows that $C(\mathbb{G}^{\mathrm{SAP}})$ is also the completion of $\mathcal{AP}(C_0(\mathbb{G}))$, namely the reduced version of $\mathbb{AP}(C_0(\mathbb{G}))$, and so the second claim follows. $\square$

Consequently, it is enough to work in $C_0(\mathbb{G})$. Combining this proposition with the observations of Section 3.2, we have the following commutative diagram, where now $\mathcal{A}$ denotes the Hopf $*$-algebra associated with $\mathbb{G}^{\mathrm{SAP}}$, which can now be identified the space of elements of admissible representations of $C_0^u(\mathbb{G})$, or equivalently, $C_0(\mathbb{G})$,

$$\begin{array}{ccccccc} C^u(\mathbb{G}^{\mathrm{SAP}}) & \twoheadrightarrow & \mathbb{AP}(C_0^u(\mathbb{G})) & \xrightarrow{\text{Restriction of } \Lambda_{\mathbb{G}}} & \mathbb{AP}(C_0(\mathbb{G})) & \twoheadrightarrow & C(\mathbb{G}^{\mathrm{SAP}}) \\ & \searrow & \uparrow & & \uparrow & \nearrow & \\ & & & \mathcal{A} & & & \end{array}$$

In Section 6.2 below, we shall see that in the cocommutative case, it is possible to say when the horizontal surjections are actually isomorphisms.

Definition 3.10. For a locally compact quantum group $\mathbb{G}$, we write $\mathcal{AP}(\mathbb{G})$ for the unique dense Hopf $*$-algebra of $C(\mathbb{G}^{\mathrm{SAP}})$. By the above, equivalently this is the unique dense Hopf $*$-algebra of $\mathbb{AP}(C_0(\mathbb{G}))$, or of $\mathbb{AP}(C_0^u(\mathbb{G}))$.

We finish this section by showing a simple link between admissible representations and the antipode S, thought of here as a strictly-closed (unbounded) operator on $M(C_0(\mathbb{G}))$.

Proposition 3.11. *Let $U = (U_{ij}) \in \mathbb{M}_n(M(C_0(\mathbb{G})))$ be a unitary corepresentation. Then U is admissible if and only if $U_{ij}^* \in D(S)$ for each i, j.*

Proof. For fixed i, j, as U is a corepresentation and is unitary,

$$\sum_k \Delta(U_{ik})(1 \otimes U_{jk}^*) = \sum_{k,l} U_{il} \otimes U_{lk} U_{jk}^* = \sum_l U_{il} \otimes (UU^*)_{lj} = U_{ij} \otimes 1.$$

Similarly,

$$\sum_k (1 \otimes U_{ik})\Delta(U_{jk}^*) = \sum_{k,l} U_{jl}^* \otimes U_{ik} U_{lk}^* = \sum_l U_{jl}^* \otimes (UU^*)_{il} = U_{ji}^* \otimes 1.$$

It follows from [29, Corollary 5.34, Remark 5.44] that $U_{ij} \in D(S)$ with $S(U_{ij}) = U_{ji}^*$.

Suppose now that $U_{ij}^* \in D(S)$ for all i, j. Then $U_{ij} \in D(S^{-1})$, so we may set $V_{ij} = S^{-1}(U_{ji})$. Then

$$\sum_k V_{ik}(U^\top)_{kj} = \sum_k S^{-1}(U_{ki}) U_{jk} = S^{-1}\Big(\sum_k S(U_{jk}) U_{ki}\Big) = S^{-1}\Big(\sum_k U_{kj}^* U_{ki}\Big) = \delta_{i,j} 1,$$

so that $VU^\top = 1$. Similarly, $U^\top V = 1$, so $U^\top$ is invertible.

Conversely, if U is admissible, then the elements of U will belong to the Hopf $*$-algebra $\mathcal{A}$ associated to $\mathbb{AP}(C_0(\mathbb{G}))$. By applying [29, Proposition 5.45] (compare [43, Proposition 4.11]) to the inclusion Hopf $*$-homomorphism $\mathbb{AP}(C_0(\mathbb{G})) \to M(C_0(\mathbb{G}))$ we see that this inclusion will intertwine the antipode on $\mathcal{A}$ and S. In particular, S will restrict to a bijection $\mathcal{A} \to \mathcal{A}$, and so the "only if" claim follows. □

Remark 3.12. Thanks to [27, Proposition 9.6], the same result holds for unitary corepresentations of $C_0^u(\mathbb{G})$ if we use the universal antipode S_u.

We also remark that the proof that matrix elements of unitary corepresentations of a compact quantum group form a Hopf $*$-algebra ultimately relies upon the fact that if U is a unitary corepresentation, then $\overline{U}$, or equivalently $U^\top$, is similar to a unitary corepresentation (equivalently, anti-corepresentation). Indeed, we used this fact in the proof of Proposition 3.9 above. This point was, we feel, slightly skipped in [43, Remark 2.3(2)] and explains why the argument given there does not work (directly) for *locally* compact quantum groups.

Remark 3.13. If $\mathbb{G}$ is a Kac algebra (or, more generally, if $S = R$) then the previous proposition gives a simple proof that any unitary corepresentation is admissible. In particular, this answers the implicit question before [43, Proposition 4.6].

Remark 3.14. Let us just remark that everything in this section applies equally well to quantum groups coming from manageable multiplicative unitaries– simply replace references to [27] by the appropriate results to be found in [46] and [34].

4 Representation free, and Banach algebraic, techniques

When G is a locally compact group, the algebra $\mathbb{AP}(C_0(G))$ coincides with the classical algebra of *almost periodic functions*, namely those functions $f \in C^b(G)$ such that the collection of left (or right) translates of f forms a relatively compact subset of $C^b(G)$, see [7, Section 4.3] for example.

There is a classical and well-studied link with Banach algebras here. Consider the algebra $L^1(G)$ and turn $C^b(G)$ into an $L^1(G)$ bimodule in the usual way; we shall denote the module actions by $\star$. Then a simple argument using the bounded approximate identity for $L^1(G)$, together with the fact that a subset of a Banach space is relatively compact if and only if its absolutely convex hull is, shows that a function f is almost periodic if and only if the orbit map $L^1(G) \to C^b(G); a \mapsto a \star f$ (or $f \star a$) is a compact linear map. In fact, identifying $C^b(G)$ with a subalgebra of $L^\infty(G)$, we obtain the same class by looking at those $f \in L^\infty(G)$ with $L^1(G) \to L^\infty(G); a \mapsto a \star f$ being compact (compare with the arguments of [49] or [9, Lemma 5.1] for example).

We are hence lead to consider the following definition.

Definition 4.1. Let $\mathfrak{A}$ be a Banach algebra, and turn $\mathfrak{A}^*$ into an $\mathfrak{A}$-bimodule in the usual way. A functional $\mu \in \mathfrak{A}^*$ is *almost periodic* if the orbit map $\mathfrak{A} \to \mathfrak{A}^*; a \mapsto a \star \mu$ is compact. We write $AP(\mathfrak{A})$ for the collection of such functionals.

For $\mu \in \mathfrak{A}^*$, define $L_\mu, R_\mu : \mathfrak{A} \to \mathfrak{A}^*$ by $R_\mu(a) = a \star \mu$ and $L_\mu(a) = \mu \star a$. With $\kappa : \mathfrak{A} \to \mathfrak{A}^{**}$ being the canonical map, we find that $L_\mu^* \circ \kappa = R_\mu$ and $R_\mu^* \circ \kappa = L_\mu$, and so L_μ is compact if and only if R_μ is compact. We remark that some authors write $AP(\mathfrak{A}^*)$ instead of $AP(\mathfrak{A})$.

When $\mathfrak{A} = L^1(G)$, we hence recover the classical notion of an almost periodic function. It is thus very tempting to use the same definition for any locally compact quantum group. Indeed, early on in the development of the Fourier Algebra, the definition of $AP(\hat{G}) = AP(A(G))$ was made (we believe for the first time in [14, Chapter 7]).

Let us quickly recall some notation. For a locally compact group G we define $L^\infty(\hat{G})$ to be the group von Neumann algebra $VN(G)$, which is generated by the left translation maps $\{\lambda(s) : s \in G\}$ acting on $L^2(G)$. In the setup of Kac algebras or locally compact quantum groups, this is the dual to the commutative algebra $L^\infty(G)$. The predual of $VN(G)$ is the Fourier algebra $A(G)$, as defined by Eymard, [19] (compare also [47,]). Generally one thinks of $A(G)$ as being a commutative Banach algebra, in fact, a subalgebra of $C_0(G)$. We remark that the locally compact quantum group convention would be to consider multiplicative unitary W for $VN(G)$, and then to consider the "left-regular representation", the map $A(G) = VN(G)_* \to C_0(G); \omega \mapsto (\omega \otimes \iota)(W)$ (which is also used in [47, Section 3, Chapter VII]). Concretely, we identify the functional ω with the continuous function $G \to \mathbb{C}; t \mapsto \langle \lambda(t^{-1}), \omega \rangle$. We warn the reader that Eymard instead considers the map $t \mapsto \langle \lambda(t), \omega \rangle$.

It was recognised that it "should be" the case that $AP(\hat{G}) = C^*_\delta(G)$, the C*-subalgebra of $VN(G)$ generated by the left translation operators. This is indeed quantum Bohr compactification, see [43, Section 4.2] and Section 5 below. An excellent reference here is Chou's paper [9].

Theorem 4.2. *We have the following facts:*

1. *$AP(\hat{G}) = C^*_\delta(G)$ if G is abelian [14], or discrete and amenable (which follows easily from [20, Proposition 2]).*

2. *there exists a compact group G such that $C^*_\delta(G) \neq AP(\hat{G})$. This is [9, Theorem 3.5], but be aware of some errors in preliminary results; these errors are partly corrected in [40]; in particular [40, Proposition 1] shows the result we are interested in.*

3. *if H is an open normal subgroup of G with G/H amenable, and with $C^*_\delta(H) = AP(\hat{H})$, then also $C^*_\delta(G) = AP(\hat{G})$, [9, Theorem 4.4].*

Remarkably, in full generality, it is still unknown if $AP(\hat{G})$ is even a C*-algebra, never-mind whether it satisfies any obvious interpretation as a "compactification". Recently Runde suggested a new definition of "almost periodic" which takes account of the Operator Space structure of $A(G)$. We use standard notions from the theory of Operator Spaces, see [17] for example. In particular, if M is a von Neumann algebra and M_* its predual, then the space of completely bounded maps $M_* \to M$, denoted $\mathcal{CB}(M_*, M)$, can be identified with the dual space of the operator space projective tensor product, $M_* \widehat{\otimes} M_*$, and with the von Neumann tensor product $M \overline{\otimes} M$. That is, we identify $T \in \mathcal{CB}(M_*, M)$, $\mu \in (M_* \widehat{\otimes} M_*)^*$ and $y \in M \overline{\otimes} M$ by the relations

$$\langle T(\omega), \tau \rangle = \langle \mu, \omega \otimes \tau \rangle = \langle y, \omega \otimes \tau \rangle \qquad (\omega, \tau \in M_*).$$

There are analogous constructions given by slicing y on the right; see [17, Chapter 7].

For a map between operator spaces, there are a number of notions of being "compact", namely *completely compact* and *Gelfand compact*, see [42, Section 1] and references therein for a discussion. In general these are distinct, but when mapping into a dual, *injective* operator space, they coincide with the notation of being the completely-bounded-norm limit of finite-rank operators (much as, in the presence of the approximation property, a compact map between Banach spaces is the norm limit of finite-rank maps). For a completely contractive Banach algebra $\mathfrak{A}$, Runde makes the following definitions:

Definition 4.3. A completely bounded map $T : E \to F$ between operator spaces is *completely compact* if for each $\epsilon > 0$ there is a finite-dimensional subspace Y of F such that, with $Q : F \to F/Y$ the quotient map, $\|QT\|_{cb} < \epsilon$.

For $\mu \in \mathcal{A}^*$ say that μ is *completely almost periodic*, denoted by $\mu \in \mathrm{CAP}(\mathfrak{A})$, if both L_μ and R_μ are completely compact.

Consider the case when (M, Δ) is a Hopf von-Neumann algebra and $\mathfrak{A} = M_*$ with the canonical operator space structure. Then, if M is an injective von Neumann algebra, [42, Theorem 2.4] shows that $\mathrm{CAP}(\mathfrak{A})$ is a C*-subalgebra of M. Indeed, the proof shows that

$$\mathrm{CAP}(\mathfrak{A}) = \{x \in M : \Delta(x) \in M \otimes M\},$$

where here $\otimes$ denotes the minimal C*-algebra tensor product, namely the norm closure of $M \odot M$ in $M \overline{\otimes} M$. In particular, this applies to $\mathfrak{A} = A(G)$ when G is an amenable or connected locally compact group, [42, Corollary 2.5].

Hence for "nice" $\mathbb{G}$, Runde's algebra $\mathrm{CAP}(L^1(\mathbb{G}))$ agrees with those $x \in L^\infty(\mathbb{G})$ such that

$$L_x : L^1(\mathbb{G}) \to L^\infty(\mathbb{G}); \quad \omega \mapsto x \star \omega = (\omega \otimes \iota)\Delta(x)$$

can be cb-norm approximated by finite-rank maps. Equivalently, this means that $\Delta(x)$ can be norm approximated by finite-rank tensors in $M \overline{\otimes} M$.

4.1 Counter-examples

The counter-example considered by Chou is as follows: for a compact group G, let E be the rank-one orthogonal projection onto the constant functions in $L^2(G)$. Then $E \in VN(G)$ and it is possible to analyse closely the orbit map $A(G) \to VN(G); \omega \mapsto \omega \cdot E$, in particular, $E \in AP(\hat{G})$ if and only if G is tall. However, a careful calculation shows that Chou's argument [9, Proposition 3.1] (compare also [15]) does extend to $\mathrm{CAP}(\hat{G})$. We plan to explore in future work if we can characterise when $E \in \mathrm{CAP}(\hat{G})$; compare also with Theorem 5.3 below.

Instead, we now turn our attention to the fully quantum case, and explore a remarkable result of Woronowicz in [59]. The quantum $E(2)$ group was defined in [60], see also [39, 53].

Theorem 4.4. *Let $q \in (0,1)$ and let $\mathbb{G}$ be the quantum $E(2)$ group with parameter q. Then* $\mathrm{CAP}(\mathbb{G})$ *strictly contains* $\mathbb{AP}(C_0(\mathbb{G})) \cong C(\mathbb{G}^{\mathrm{SAP}})$.

Proof. We use two results of Woronowicz. Firstly, [59] (see especially Section 4) shows that for all $x \in C_0(\mathbb{G})$, we have that $\Delta(x) \in C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$; notice the lack of a multiplier algebra! So immediately we see that $C_0(\mathbb{G}) \subseteq \mathrm{CAP}(\mathbb{G})$.

Secondly, we use the classification of unitary corepresentations of $C_0(\mathbb{G})$ given in [60, Theorem 2.1]. In finite-dimensions, the classification is very restrictive. Indeed, let $U \in M(C_0(\mathbb{G})) \otimes \mathbb{M}_n$ be a finite-dimensional unitary corepresentation. Then there exist matrices $N, b \in \mathbb{M}_n$ with N self-adjoint, b normal, and $N, |b|$ commuting. Furthermore, if b has polar decomposition $b = u|b|$, then on $(\ker b)^\perp$, we have that $u^* N u = N + 2$. Clearly this cannot hold for *bounded* operators, so in this finite-dimensional setting, $(\ker b)^\perp = \{0\}$ so $b = 0$. Then [60, Theorem 2.1] further gives an expression for U in terms of b and N. However, as $b = 0$, it follows that actually $U \in \mathbb{M}_n(M(C_0(\mathbb{G})))$ diagonalises, with diagonal entries powers of $v \in M(C_0(\mathbb{G}))$. Here v is a unitary, one of the operators which "generates" $C_0(\mathbb{G})$, compare [60, Theorem 1.1]. We know that $\Delta(v) = v \otimes v$ by [60, Theorem 1.2], and so v is a one-dimensional, admissible, corepresentation.

It follows from this discussion that $\mathcal{AP}(\mathbb{G})$ is spanned by $\{v^k : k \in \mathbb{Z}\}$ and so $\mathbb{G}^{\mathrm{SAP}}$ is isomorphic to the circle group (and hence is a classical compact group). In particular, $\mathbb{AP}(C_0(\mathbb{G}))$ is (much) smaller than $\mathrm{CAP}(\mathbb{G})$. □

Let us draw one further conclusion from this. Let $A \subseteq M(C_0(\mathbb{G}))$ be the maximal compact quantum semigroup; so A is the maximal unital C*-subalgebra with $\Delta(A) \subseteq A \otimes A$. That A exists follows from a free-product argument, compare [54]: if $B, C \subseteq M(C_0(\mathbb{G}))$ are two such unital algebras, then the image of the free-product $B * C$ in $M(C_0(\mathbb{G}))$ will be a unital C*-bialgebra containing both B and C. If $\mathbb{G} = G$ is a classical (semi)group then A will be commutative, with character space K say, and Δ restricted to A will induce a continuous semigroup structure on K. It follows that K is actually the "almost periodic" compactification of G, compare [7, Chapter 4.1]. If G were a group, then this agrees with the Bohr compactification. However, if now $\mathbb{G}$ is again the quantum $E(2)$ group, then this maximal A will certainly contain $C_0(\mathbb{G}) \oplus \mathbb{C}1$; in particular, again A will be (much) larger than $\mathbb{AP}(C_0(\mathbb{G}))$.

4.2 Stronger notions

Thus it appears that, in complete generality, even the notion of "completely almost periodic" is not strong enough to single out $\mathbb{AP}(C_0(\mathbb{G}))$. We shall make a stronger definition; the link with Runde's definition is clarified in Proposition 4.9 below.

Definition 4.5. Say that $x \in L^\infty(\mathbb{G})$ is *periodic* if $\Delta(x)$ is a finite-rank tensor in $L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$. Denote the collection of periodic elements of $L^\infty(\mathbb{G})$ by $\mathcal{P}^\infty(\mathbb{G})$, and denote the norm closure, in $L^\infty(\mathbb{G})$, by $\mathbb{P}^\infty(\mathbb{G})$.

Note that similar definitions would hold for $M(C_0(\mathbb{G}))$ (or indeed for $M(C_0^u(\mathbb{G}))$) and at this level of generality, it is not clear if they would yield the same objects; thus we choose to emphasis this by using the notation $\mathcal{P}^\infty$ not $\mathcal{P}$, and so forth. Compare with Remark 4.15 and the comment after Theorem 6.4, which suggest that working with $M(C_0(\mathbb{G}))$ should probably give the same result, but that $M(C_0^u(\mathbb{G}))$ might not.

Remark 4.6. In the purely algebraic setting, working with multiplier Hopf algebras, Sołtan studied a very similar ideas for discrete quantum groups in [44] (see also [43, Proposition 4.6]).

Lemma 4.7. *Let $\mathbb{G}$ be a locally compact quantum group. Then*

$$\mathcal{P}^\infty(\mathbb{G}) = \{x \in L^\infty(\mathbb{G}) : L_x \text{ is finite-rank}\} = \{x \in L^\infty(\mathbb{G}) : R_x \text{ is finite-rank}\}.$$

Furthermore, $\mathbb{P}^\infty(\mathbb{G})$ is a C-subalgebra of $L^\infty(\mathbb{G})$, and an $L^1(\mathbb{G})$-submodule of $L^\infty(\mathbb{G})$.*

Proof. It follows immediately from the definition that $\mathbb{P}^\infty(\mathbb{G})$ is a C*-algebra. As $L_x(\omega) = x \star \omega = (\omega \otimes \iota)\Delta(x)$, it is easy to see that if $\Delta(x)$ is a finite-rank tensor, then L_x is a finite-rank map. Conversely, if L_x is finite-rank, then let $\{x_i\}$ be a basis for the image of L_x. For each $\omega \in L^1(\mathbb{G})$, there hence exist unique scalars $\{a_i\}$ with $x \star \omega = \sum_i a_i x_i$. Then the map $\omega \mapsto a_i$ is bounded and linear, so there are y_i in $L^\infty(\mathbb{G})$ with $x \star \omega = \sum_i \langle y_i, \omega\rangle x_i$. Equivalently, $\Delta(x) = \sum_i y_i \otimes x_i$ is a finite-rank tensor. An analogous argument holds for R_x; or use again that $R_x = L_x^* \circ \kappa$ and $L_x = R_x^* \circ \kappa$ (where κ is the inclusion $L^1(\mathbb{G}) \to L^1(\mathbb{G})^{**}$).

To show that $\mathbb{P}^\infty(\mathbb{G})$ is an $L^1(\mathbb{G})$-submodule, it suffices to show that $\mathcal{P}^\infty(\mathbb{G})$ is a submodule. For $\omega \in L^1(\mathbb{G})$, let $T_\omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ be the map $x \mapsto \omega \star x$. Then $L_{\omega\star x} = T_\omega \circ L_x$, so $x \in \mathcal{P}^\infty(\mathbb{G}) \implies \omega \star x \in \mathcal{P}^\infty(\mathbb{G})$. Similarly, let $S_\omega : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ be the map $\tau \mapsto \omega\tau$. Then $L_{x\star\omega} = L_x \circ S_\omega$, and it follows that $\mathcal{P}^\infty(\mathbb{G})$ is also a right $L^1(\mathbb{G})$-submodule. ☐

The following ultimately provides an alternative description of $\mathcal{P}^\infty(\mathbb{G})$.

Theorem 4.8. *Let $T : L^1(\mathbb{G}) \to L^\infty(\mathbb{G})$ be a completely bounded right $L^1(\mathbb{G})$-module homomorphism. Then there exists $x \in L^\infty(\mathbb{G})$ with $T = R_x$.*

Proof. That T is completely bounded again means that there is $y \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ with $T(\omega) = (\omega \otimes \iota)(y)$ for all $\omega \in L^1(\mathbb{G})$. If we have that $y = \Delta(x)$, then $T(\omega) = (\omega \otimes \iota)\Delta(x) = \omega \star x = R_x(\omega)$ as required. Conversely, that T is a right $L^1(\mathbb{G})$-module homomorphism is equivalent to

$$\langle y, \omega_1\omega_2 \otimes \omega_3\rangle = \langle T(\omega_1\omega_2), \omega_3\rangle = \langle T(\omega_1), \omega_2\omega_3\rangle = \langle y, \omega_1 \otimes \omega_2\omega_3\rangle,$$

that is, $(\Delta \otimes \iota)(y) = (\iota \otimes \Delta)(y)$. So we wish to prove that this relation on y forces that $y = \Delta(x)$ for some x.

If $L^1(\mathbb{G})$ has a bounded approximate identity, this is a Banach algebraic exercise. For $\mathbb{G}$ cocommutative, we gave a proof in [11, Theorem 6.5], and an early preprint by the author[2] shows that this is true for general $\mathbb{G}$ (with a relatively elementary proof). However, actually a somewhat more general statement already exists in the literature. A *(left) action* of $\mathbb{G}$ on a von Neumann algebra N is an injective normal unital $*$-homomorphism $\alpha : N \to L^\infty(\mathbb{G})\overline{\otimes}N$ with $(\iota\otimes\alpha)\alpha = (\Delta\otimes\iota)\alpha$. For actions of Kac algebras, it was shown in [18, Théorème IV.2] (see also [18, Définition II.8]) that

$$\alpha(N) = \{z \in L^\infty(\mathbb{G})\overline{\otimes}N : (\iota \otimes \alpha)(z) = (\Delta \otimes \iota)(z)\}.$$

In [50, Section 2] the necessary preliminary steps to prove this result for locally compact quantum groups are given, although a full proof is not shown. As Δ is an action of $\mathbb{G}$ on $L^\infty(\mathbb{G})$, our claim immediately follows from this general theory. ☐

[2] See Theorem 2.2 of `arXiv:1107.5244v3 [math.OA]`

Proposition 4.9. *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{AP}(C_0(\mathbb{G})) \subseteq \mathbb{P}^\infty(\mathbb{G}) \subseteq \mathrm{CAP}(L^1(\mathbb{G}))$. Indeed, $x \in \mathbb{P}^\infty(\mathbb{G})$ if and only if L_x can be cb-norm approximated by finite-rank module maps $L^1(\mathbb{G}) \to L^\infty(\mathbb{G})$.*

Proof. It is clear that a matrix element of an admissible corepresentation is periodic (as the corepresentation is finite-dimensional) and so $\mathbb{AP}(C_0(\mathbb{G})) \subseteq \mathbb{P}^\infty(\mathbb{G})$. If x is periodic, then as Δ is an isometry, it follows that $\Delta(x)$ can be norm approximated by elements of form $\Delta(y)$ with $\Delta(y)$ a finite-rank tensor. It is immediate that $x \in \mathrm{CAP}(L^1(\mathbb{G}))$, and that L_x can be cb-norm approximated by finite-rank module maps. The converse now follows from the previous theorem, as every finite-rank module map is of the form L_y for some y with $\Delta(y)$ a finite-rank tensor. □

Thus the collection $\mathbb{P}^\infty$, which can be defined purely in terms of the Banach algebra $L^1(\mathbb{G})$, is a weakening of $\mathbb{AP}$ and a strengthening of CAP. In the next section we shall provide cases (in particularly, when $\mathbb{G}$ is a Kac algebra) when $\mathbb{AP} = \mathbb{P}^\infty$. In the following section we embark on a programme to determine $\mathrm{CAP}(A(G))$ for various classes of groups G.

4.3 When periodic implies almost periodic

Our aim here is to show that if $x \in \mathbb{P}^\infty(\mathbb{G})$, then under further assumptions on x, also $x \in \mathbb{AP}(C_0(\mathbb{G}))$. Firstly, we need to decide upon reasonable "further assumptions". Proposition 3.11 immediately implies the following.

Lemma 4.10. *Let $x \in \mathcal{AP}(\mathbb{G})$. If we consider $\mathcal{AP}(\mathbb{G})$ as a subalgebra of $M(C_0(\mathbb{G}))$, then $x \in D(S) \cap D(S^{-1})$. Similarly, if we consider $\mathcal{AP}(\mathbb{G}) \subseteq M(C_0^u(\mathbb{G}))$, then $\mathcal{AP}(\mathbb{G}) \subseteq D(S_u) \cap D(S_u^{-1})$.*

Our aim will be to show that in fact $\mathcal{AP}(\mathbb{G}) = \mathbb{P}^\infty(\mathbb{G}) \cap D(S) \cap D(S^{-1})$. Notice that if S is actually bounded (if $\mathbb{G}$ is a Kac algebra) then immediately we have that $\mathcal{AP}(\mathbb{G}) = \mathbb{P}(\mathbb{G})$.

We start with some results about the antipode. As S is unbounded, the natural candidate for an involution on $L^1(\mathbb{G})$ is unbounded. Instead, following for example [27, Section 4], we define $L^1_\sharp(\mathbb{G})$ to be the collection of $\omega \in L^1(\mathbb{G})$ such that there is $\omega^\sharp \in L^1(\mathbb{G})$ satisfying $\langle x, \omega^\sharp \rangle = \overline{\langle S(x)^*, \omega \rangle}$ for all $x \in D(S)$. Then $\omega \mapsto \omega^\sharp$ defines an involution on $L^1_\sharp(\mathbb{G})$. For $\omega \in L^1(\mathbb{G})$ define ω^* by $\langle x, \omega^* \rangle = \overline{\langle x^*, \omega \rangle}$ for $x \in L^\infty(\mathbb{G})$.

Lemma 4.11. *Let $x \in D(S) \subseteq L^\infty(\mathbb{G})$ and let $\omega \in L^1_\sharp(\mathbb{G})$. Then $x \star \omega \in D(S)$ with $S(x \star \omega) = \omega^{\sharp *} \star S(x)$, and $\omega \star x \in D(S)$ with $S(\omega \star x) = S(x) \star \omega^{\sharp *}$.*

Let $y \in D(S^{-1})$ and let $\tau \in L^1_\sharp(\mathbb{G})^*$. Then $y \star \tau, \tau \star y \in D(S^{-1})$ with $S^{-1}(y \star \tau) = \tau^{*\sharp} \star S^{-1}(y)$ and $S^{-1}(\tau \star y) = S^{-1}(y) \star \tau^{*\sharp}$.

Proof. By [3, Appendix A], for example, to show that $x \star \omega \in D(S)$ with $S(x \star \omega) = \omega^{\sharp *} \star S(x)$, it suffices to show that for all $\tau \in L^1_\sharp(\mathbb{G})$, we have that

$$\langle x \star \omega, \tau^\sharp \rangle = \langle \omega^{\sharp *} \star S(x), \tau^* \rangle.$$

However, this follows from a simple calculation:

$$\langle x \star \omega, \tau^\sharp \rangle = \langle x, \omega \star \tau^\sharp \rangle = \langle x, (\tau \star \omega^\sharp)^\sharp \rangle = \overline{\langle S(x)^*, \tau \star \omega^\sharp \rangle} = \overline{\langle \omega^\sharp \star S(x)^*, \tau \rangle} = \langle \omega^{\sharp *} \star S(x), \tau^* \rangle.$$

The second claim is entirely analogous.

For the second part, we use that $S^{-1} = * \circ S \circ *$, and that Δ is a $*$-homomorphism. So $y^* \in D(S)$ and $\tau^* \in L^1_\sharp(\mathbb{G})$, and thus $y^* \star \tau^* \in D(S)$ with $S(y^* \star \tau^*) = \tau^{*\sharp *} \star S(y^*)$. Hence $y \star \tau \in D(S^{-1})$ and $S^{-1}(y \star \tau) = S(y^* \star \tau^*)^* = \tau^{*\sharp} \star S^{-1}(y)$. The other claim follows similarly. □

The following is essentially a restatement of [3, Proposition 4.6]; but here our sums are finite, and so we can ignore convergence issues.

Proposition 4.12. *Let $x \in L^\infty(\mathbb{G})$ be such that $\Delta(x) = \sum_{i=1}^n a_i \otimes b_i$ where for each i, we have that $b_i^* \in D(S)$. Then the map $L : L^\infty(\hat{\mathbb{G}}) \to \mathcal{B}(L^2(\mathbb{G})); \hat{x} \mapsto \sum_{i=1}^n S(b_i^*)^* \hat{x} a_i$ maps into $L^\infty(\hat{\mathbb{G}})$ and is the adjoint of a (completely bounded) left multiplier of $L^1(\hat{\mathbb{G}})$. In particular, $(\iota \otimes L)(W^*) = (x \otimes 1)W^*$, equivalently, $\sum_i (1 \otimes S(b_i^*)^*)\Delta(a_i) = x \otimes 1$.*

Corollary 4.13. *Let $x \in L^\infty(\mathbb{G})$ be such that $\Delta(x) = \sum_{i=1}^n a_i \otimes b_i$ where for each i, we have that $a_i \in D(S)$. Then $1 \otimes x = \sum_i (S(a_i) \otimes 1)\Delta(b_i)$.*

Proof. We follow [30, Section 4], and define the *opposite quantum group* to $\mathbb{G}$ to be $\mathbb{G}^{\mathrm{op}}$, where $L^\infty(\mathbb{G}^{\mathrm{op}}) = L^\infty(\mathbb{G})$ and $\Delta^{\mathrm{op}} = \sigma\Delta$. Then we find that $R^{\mathrm{op}} = R$ and $(\tau_t^{\mathrm{op}}) = (\tau_{-t})$. It follows that $S^{\mathrm{op}} = R^{\mathrm{op}}\tau^{\mathrm{op}}_{-i/2} = R\tau_{i/2} = S^{-1} = * \circ S \circ *$.

So $\Delta^{\mathrm{op}}(x) = \sum_i b_i \otimes a_i$ and for each i, we have that $a_i^* \in D(S^{\mathrm{op}})$. So the previous proposition, now applied to $\mathbb{G}^{\mathrm{op}}$, shows that $x \otimes 1 = \sum_i (1 \otimes S^{\mathrm{op}}(a_i^*)^*)\Delta^{\mathrm{op}}(b_i)$, or equivalently, $1 \otimes x = \sum_i (S(a_i) \otimes 1)\Delta(b_i)$. ☐

We can now prove the main theorem of this section; the idea of this construction is well-known in Hopf algebra theory (but of course we have to work harder in the analytic setting).

Theorem 4.14. *Let $x \in \mathcal{P}^\infty(\mathbb{G}) \cap D(S) \cap D(S^{-1})$. There exists an admissible corepresentation $U = (U_{ij})$ of $C_0(\mathbb{G})$ such that x is a matrix element of U. In particular, $x \in M(C_0(\mathbb{G}))$ and $x \in \mathcal{AP}(C_0(\mathbb{G}))$.*

Proof. That $x \in \mathcal{P}^\infty(\mathbb{G})$ is equivalent to $R_x : L^1(\mathbb{G}) \to L^\infty(\mathbb{G}); \omega \mapsto \omega \star x$ having a finite-dimensional image, say X. Then $R_x(L^1_\sharp(\mathbb{G}))$ is dense in X, as $L^1_\sharp(\mathbb{G})$ is dense in $L^1(\mathbb{G})$. As X is finite-dimensional, $R_x(L^1_\sharp(\mathbb{G})) = X$, and so we can find $(\omega_i)_{i=1}^n \subseteq L^1_\sharp(\mathbb{G})$ with $\{\omega_i \star x : 1 \leq i \leq n\}$ forming a basis for X. Set $x_i = \omega_i \star x$.

As $\{x_i\}$ is a linearly independent set, the map $L^1(\mathbb{G}) \to \mathbb{C}^n; \omega \mapsto (\langle x_i, \omega\rangle)_{i=1}^n$ is a linear surjection. Hence the set $\{(\langle x_i, \omega\rangle)_{i=1}^n : \omega \in L^1_\sharp(\mathbb{G})\}$ is a dense linear subspace of $\mathbb{C}^n$, and hence equals $\mathbb{C}^n$. So we can find $(\tau_i)_{i=1}^n \subseteq L^1_\sharp(\mathbb{G})$ with $\langle x_j, \tau_i\rangle = \delta_{i,j}$ for $1 \leq i, j \leq n$. Let $y_i = x \star \tau_i$.

For $\omega \in L^1(\mathbb{G})$ there are unique $(a_i)_{i=1}^n \subseteq \mathbb{C}$ with $\omega \star x = \sum_i a_i x_i$. Then $a_i = \langle \omega \star x, \tau_i\rangle = \langle x \star \tau_i, \omega\rangle = \langle y_i, \omega\rangle$, and so for $\tau \in L^1(\mathbb{G})$,

$$\langle \Delta(x), \tau \otimes \omega\rangle = \sum_i a_i \langle x_i, \tau\rangle = \sum_i \langle x_i \otimes y_i, \tau \otimes \omega\rangle.$$

It follows that $\Delta(x) = \sum_i x_i \otimes y_i$. Notice that then $x_i = \omega_i \star x = \sum_j \langle y_j, \omega_i\rangle x_j$, so as $\{x_i\}$ is a linearly independent set, we conclude that $\langle y_j, \omega_i\rangle = \delta_{i,j}$ for all i, j. In particular, $\{y_j\}$ is also a linearly independent set.

Set $U_{ij} = x_j \star \tau_i = \omega_j \star x \star \tau_i$. Then

$$\Delta^2(x) = \sum_j \Delta(x_j) \otimes y_j = \sum_i x_i \otimes \Delta(y_i).$$

It follows that $\Delta(y_i) = \sum_j U_{ij} \otimes y_j$ for all i. Then

$$\Delta^2(y_i) = \sum_j \Delta(U_{ij}) \otimes y_j = \sum_k U_{ik} \otimes \Delta(y_k) = \sum_{k,j} U_{ik} \otimes U_{kj} \otimes y_j.$$

As $\{y_j\}$ is also a linearly independent set, this shows that $U = (U_{ij})$ is a corepresentation.

Using Lemma 4.11, notice that $y_i, x_i \in D(S)$ for all i, and that $U_{ij} \in D(S)$ for all i, j. As $\Delta(x) = \sum_i x_i \otimes y_i$, Corollary 4.13 shows that

$$1 \otimes x = \sum_i (S(x_i) \otimes 1)\Delta(y_i) = \sum_{i,j} S(x_i)U_{ij} \otimes y_j.$$

Hence $x \in \lin\{y_i\}$. We now notice that for each i, $\Delta(y_i^*) = \sum_j U_{ij}^* \otimes y_j^*$, and so Proposition 4.12 shows that

$$y_i^* \otimes 1 = \sum_j (1 \otimes S(y_j)^*)\Delta(U_{ij}^*) = \sum_{jk} U_{ik}^* \otimes S(y_j)^* U_{kj}^*.$$

Hence $y_i \in \lin\{U_{ik} : 1 \leq k \leq n\}$ for each i, and we conclude that x is a matrix element of U.

The preceding argument is partly inspired by [3, Section 4]; in particular, [3, Theorem 4.7] shows that if the representation $L^1(\mathbb{G}) \to \mathbb{M}_n; \omega \mapsto (\langle U_{ij}, \omega\rangle)$ is non-degenerate, then U is invertible with inverse $(S(U_{ij}))_{i,j=1}^n$. We claim that this representation is indeed non-degenerate. For $\xi \in \mathbb{C}^n$, as $\{y_i\}$ is a linearly independent set, there is $\tau \in L^1(\mathbb{G})$ with $\xi = (\langle y_j, \tau\rangle)_{j=1}^n$. Then

$$(\langle U_{ij}, \omega\rangle)\xi = \Big(\sum_j \langle U_{ij}, \omega\rangle\langle y_j, \tau\rangle\Big)_{i=1}^n = \big(\langle y_i, \omega \star \tau\rangle\big)_{i=1}^n.$$

We need to show that as ω, τ vary, we get a subset whose linear span is all of $\mathbb{C}^n$. However, this is clear, because $\lin\{\omega \star \tau : \omega, \tau \in L^1(\mathbb{G})\}$ is a dense subspace of $L^1(\mathbb{G})$, and the set $\{y_i\}$ is linearly independent.

We hence conclude that U is invertible. It is known that invertible representations are members of $M(C_0(\mathbb{G}) \otimes \mathbb{M}_n) = M(C_0(\mathbb{G})) \otimes \mathbb{M}_n$, see [57, Section 4], [2, Page 441] and [3, Theorem 4.9]. So $x \in M(C_0(\mathbb{G}))$.

It remains to show that U is admissible, that is, $U^\top$ is invertible, or equivalently, that $\overline{U} = (U_{ij}^*)_{i,j=1}^n$ is invertible. By hypothesis, also $x \in D(S^{-1})$. Arguing as before, we can find $(\omega_i')_{i=1}^n \subseteq L^1_\natural(\mathbb{G})^*$ with $x_i = \omega_i' \star x$. Similarly, we find $(\tau_i')_{i=1}^n \subseteq L^1_\natural(\mathbb{G})^*$ with $\langle x_j, \tau_i'\rangle = \delta_{ij}$. Setting $y_i' = x \star \tau_i'$, we follow the argument above to conclude that $\Delta(x) = \sum_i x_i \otimes y_i'$. As $\{x_i\}$ is a linearly independent set, actually $y_i' = y_i$ for all i. Then $U_{ij}' = x_j \star \tau_i'$ is a corepresentation, but now $U_{ij}' \in D(S^{-1})$ for all i, j. However,

$$\sum_j U_{ij} \otimes y_j = \Delta(y_i) = \Delta(y_i') = \sum_j U_{ij}' \otimes y_j' = \sum_j U_{ij}' \otimes y_j,$$

so using that $\{y_j\}$ is a linearly independent set, it follows that $U_{ij} = U_{ij}'$ for all i, j. Thus $U_{ij}^* \in D(S)$ for all i, j, and so [3, Theorem 4.7] shows that $\overline{U}$ is invertible (as obviously $L^1(\mathbb{G}) \to \mathbb{M}_n; \omega \mapsto (\langle U_{ij}^*, \omega\rangle)_{i,j=1}^n = (\overline{\langle U_{ij}, \omega^*\rangle})_{i,j=1}^n$ is non-degenerate). □

Remark 4.15. An immediate corollary of the proof is that if $x \in \mathcal{P}^\infty(\mathbb{G}) \cap D(S)$, then $x \in M(C_0(\mathbb{G}))$ and x is a matrix element of an invertible, finite-dimensional corepresentation of $C_0(\mathbb{G})$.

Combining Lemma 4.10 and Theorem 4.14 immediately gives the following.

Corollary 4.16. *We have that* $\mathcal{AP}(C_0(\mathbb{G})) = \mathcal{P}^\infty(\mathbb{G}) \cap D(S) \cap D(S^{-1})$.

For Kac algebras, this takes on a very pleasing form, because $S = R$ is bounded. In particular, this answers a question ask by Sołtan, see [43, Page 1260], as to whether a finite-dimensional, unitary corepresentation of a discrete Kac algebra is automatically admissible– the answer is "yes", as would be true for any Kac algebra, and any finite-dimensional corepresentation.

Corollary 4.17. *Let* $\mathbb{G}$ *be a Kac algebra. Then* $\mathcal{AP}(C_0(\mathbb{G})) = \mathcal{P}^\infty(\mathbb{G})$, *and* $\mathbb{AP}(C_0(\mathbb{G})) = \mathbb{P}^\infty(\mathbb{G})$.

So at least for a Kac algebra, this corollary provides a way, just starting from $L^1(\mathbb{G})$, of finding $\mathcal{AP}(C_0(\mathbb{G}))$ and $\mathbb{AP}(C_0(\mathbb{G}))$. In the general case, $D(S) \cap D(S^{-1})$ will be smaller than $L^\infty(\mathbb{G})$, and so in principle it might be hard to calculate $\mathcal{P}^\infty(\mathbb{G}) \cap D(S) \cap D(S^{-1})$. Compare with Theorem 6.4 and Conjecture 7.1 below.

5 The cocommutative case

Sołtan showed in [43, Section 4.2] that if G is a locally compact group and $\mathbb{G} = \hat{G}$, then $\mathbb{AP}(C_0^u(\mathbb{G})) = \mathbb{AP}(C^*(G))$ is the closed linear span of the translation operators $\{L_s : s \in G\}$ inside $M(C^*(G))$. Actually, much the same proof work for $C_r^*(G) = C_0(\mathbb{G})$. So in light of Proposition 3.9, we see that if G_d denotes G with the discrete topology, then the compactification of $\mathbb{G}$ in LCQG is $\widehat{G_d}$

Let us give a different, short proof of this, using our previous results. In the literature (see [9] for example) it is common to write $C_\delta^*(G)$ for the C*-subalgebra of $VN(G) = L^\infty(\mathbb{G})$ generated by the left-translation maps $\{\lambda(s) : s \in G\}$. Let us write $\mathbb{C}[G]$ for the algebra (not norm closed) generated by the operators $\lambda(s)$. We first note that each $\lambda(s) \in M(C_0(\mathbb{G}))$, and that Δ restricts to $\mathbb{C}[G]$ turning $(\mathbb{C}[G], \Delta)$ into a unital bialgebra. Thus $(C_\delta^*(G), \Delta)$ is a unital C*-bialgebra. It is easy to verify the density conditions to show that $(C_\delta^*(G), \Delta)$ is a compact quantum group. Hence $C_\delta^*(G) \subseteq \mathbb{AP}(C_0(\mathbb{G}))$. However, Chou showed in [9, Proposition 2.3] (translated to our terminology) that $\mathcal{P}^\infty(\mathbb{G}) = \mathbb{C}[G]$, from which it follows that $\mathbb{P}^\infty(\mathbb{G}) = C_\delta^*(G)$. So, with reference to Proposition 4.9, it follows that $\mathbb{AP}(C_0(\mathbb{G})) = C_\delta^*(G)$.

5.1 Completely almost periodic elements

We now wish to investigate Runde's definition $\mathrm{CAP}(A(G))$. Firstly, we have the following result, which holds for any compact Kac algebra.

Theorem 5.1. *Let $\mathbb{G}$ be a compact Kac algebra. If $x \in L^\infty(\mathbb{G})$ with $\Delta(x) \in L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ (the C*-algebraic minimal tensor product) then $x \in C(\mathbb{G})$.*

Proof. As $\mathbb{G}$ is Kac, we have that $R = S = S^{-1}$ and the Haar state φ is a trace. Let $\xi_0 \in L^2(\mathbb{G})$ be the cyclic vector for φ. Then, for $y, z \in L^\infty(\mathbb{G})$ and $a, b \in C(\mathbb{G})$,

$$\begin{aligned}&\big((\varphi \otimes \iota)(W(y \otimes z)W^*)a\xi_0\big|b\xi_0\big) = \big((y \otimes z)W^*(\xi_0 \otimes a\xi_0)\big|W^*(\xi_0 \otimes b\xi_0)\big)\\ &= \big(\Delta(b^*)(y \otimes z)\Delta(a)\xi_0 \otimes \xi_0\big|\xi_0 \otimes \xi_0\big) = (\varphi \otimes \varphi)\big(\Delta(b^*)(y \otimes z)\Delta(a)\big)\\ &= (\varphi \otimes \varphi)\big(\Delta(ab^*)(y \otimes z)\big).\end{aligned}$$

using that φ is a trace. By [56, Theorem 2.6(4)] we know that

$$R\big((\varphi \otimes \iota)\big((ab^* \otimes 1)\Delta(y)\big)\big) = (\varphi \otimes \iota)\big(\Delta(ab^*)(y \otimes 1)\big).$$

Hence we get

$$\varphi\big(R\big((\varphi \otimes \iota)\big((ab^* \otimes 1)\Delta(y)\big)\big)z\big) = \varphi\big(R(z)\big((\varphi \otimes \iota)\big((ab^* \otimes 1)\Delta(y)\big)\big)\big),$$

using that $\varphi \circ R = \varphi$. This is then equal to

$$\begin{aligned}&(\varphi \otimes \varphi)\big((ab^* \otimes R(z))\Delta(y)\big) = \big(W^*(\xi_0 \otimes y\xi_0)\big|ba^*\xi_0 \otimes R(z^*)\xi_0\big)\\ &= \big((\iota \otimes \hat{\omega})(W^*)\xi_0\big|ba^*\xi_0\big) = \varphi\big(ab^*c\big) = \varphi\big(b^*ca\big) = \big(ca\xi_0\big|b\xi_0\big),\end{aligned}$$

where $\hat{\omega} = \omega_{y\xi_0, R(z^*)\xi_0} \in L^1(\hat{\mathbb{G}})$ and $c = (\iota \otimes \hat{\omega})(W^*) \in C(\mathbb{G})$. As $a, b \in C(\mathbb{G})$ were arbitrary, this shows that

$$(\varphi \otimes \iota)(W(y \otimes z)W^*) \in C(\mathbb{G}).$$

We remark that a similar idea to this construction, in a very different context, appears in [41, Section 3].

For $\epsilon > 0$ we can find $\tau \in L^\infty(\mathbb{G}) \odot L^\infty(\mathbb{G})$ with $\|\Delta(x) - \tau\| < \epsilon$. Thus

$$\|(\varphi \otimes \iota)(W\Delta(x)W^*) - (\varphi \otimes \iota)(W\tau W^*)\| < \epsilon.$$

However, as $W^*\Delta(x)W = 1 \otimes x$, this shows that there is some $d \in C(\mathbb{G})$ with $\|x - d\| < \epsilon$. So $x \in C(\mathbb{G})$ as required. □

Corollary 5.2. *Let $\mathbb{G}$ be a compact Kac algebra with $L^\infty(\mathbb{G})$ injective (for example, $\mathbb{G} = \hat{G}$ for a discrete amenable group G). Then $\mathrm{CAP}(L^1(\mathbb{G})) = C(\mathbb{G})$.*

To deal with the non-injective case would presumably require a much more detailed study of the notion of a completely compact map. We do not currently see how to adapt our ideas to the non-Kac setting.

In the rest of this section, we start a programme of extending Theorem 5.1. Having a trace seemed very important to this proof, so to make progress in the non-compact case we shall restrict ourselves to the case when $\mathbb{G} = \hat{G}$ for a [SIN] group G. We shall prove the following result:

Theorem 5.3. *Let G be a [SIN] group, and let $x_0 \in VN(G)$ be such that $\Delta^2(x_0) \in VN(G) \otimes VN(G) \otimes VN(G)$. Then $x_0 \in C^*_\delta(G)$.*

By definition, G is [SIN] group if G contains arbitrarily small neighbourhoods of the identity which are invariant under inner automorphisms. Equivalently, by [48, Proposition 4.1], $VN(G)$ is a finite von Neumann algebra. Discrete, compact and abelian groups are all [SIN] groups. A connected group G is [SIN] if and only if the quotient of G by its centre is compact, if and only if $G \to G^{\mathrm{SAP}}$ is injective, see [37, Chapter 12]. Also we note that [SIN] groups are always unimodular.

Recall that the fundamental unitary W of $VN(G)$ satisfies $W\xi(s,t) = \xi(ts,t)$ for $\xi \in L^2(G \times G), s, t \in G$. As $A(G)$ is commutative, we of course have that $\omega \star x = x \star \omega$ for $\omega \in A(G)$ and $x \in VN(G)$. We write the module action of $VN(G)$ on $A(G)$ by juxtaposition.

Lemma 5.4. *Let G be unimodular, and let $\xi \in L^2(G)$ satisfy $\xi(ab) = \xi(ba)$ for (almost) every $a, b \in G$. For all $z \in VN(G \times G)$ we have that $(\omega_\xi \otimes \iota)(WzW^*) \in VN(G)$. Furthermore, for $x, y \in VN(G)$ we have that $(\omega_\xi \otimes \iota)(W(x \otimes y)W^*) = \varphi_x \star y$ where $\varphi_x = (\omega_\xi x) \circ R \in A(G)$.*

Proof. Let $s, t \in G$ and let $w = (\omega_\xi \otimes \iota)(W(\lambda(s) \otimes \lambda(t))W^*)$. Let $f, g \in L^2(G)$, and calculate:

$$\begin{aligned}
\big(wf\big|g\big) &= \big((\lambda(s) \otimes \lambda(t))W^*(\xi \otimes f)\big|W^*(\xi \otimes g)\big) \\
&= \int_G \int_G W^*(\xi \otimes f)(s^{-1}a, t^{-1}b)\overline{W^*(\xi \otimes g)(a,b)}\, da\, db \\
&= \int_G \int_G \xi(b^{-1}ts^{-1}a)\overline{\xi(b^{-1}a)}\, da\; f(t^{-1}b)\overline{g(b)}\, db \\
&= \int_G \int_G \xi(ts^{-1}ab^{-1})\overline{\xi(ab^{-1})}\, da\; f(t^{-1}b)\overline{g(b)}\, db \qquad \text{by assumption on } \xi \\
&= \int_G \int_G \xi(ts^{-1}a)\overline{\xi(a)}\, da\; f(t^{-1}b)\overline{g(b)}\, db \qquad \text{as } G \text{ is unimodular} \\
&= \langle \lambda(st^{-1}), \omega_\xi \rangle \big(\lambda(t)f\big|g\big).
\end{aligned}$$

Thus $w = \langle \lambda(st^{-1}), \omega_\xi \rangle \lambda(t) \in VN(G)$. Now,

$$\varphi_{\lambda(s)} \star \lambda(t) = \langle \lambda(t), \varphi_{\lambda(s)} \rangle \lambda(t) = \langle \lambda(t^{-1}), \omega_\xi \lambda(s) \rangle \lambda(t) = w,$$

and so the stated formula holds when $x = \lambda(s)$ and $y = \lambda(t)$. By separate weak*-continuity, the formula holds for all x, y. Then, by weak*-continuity again, it follows that for any $z \in VN(G \times G)$ we do indeed have that $(\omega_\xi \otimes \iota)(WzW^*) \in VN(G)$. □

The obvious use of this result is that $(\omega_\xi \otimes \iota)(W\Delta(x)W^*) = x$ for any $x \in VN(G)$ and any suitable ξ. Thus this is very similar to the argument in Theorem 5.1 above.

Note that $\varphi_x = R(x)(\omega_\xi \circ R)$ and that if ξ is real-valued, then $\omega_\xi \circ R = \omega_\xi$, as for $x \in VN(G)$, we have that $R(x) = \hat{J}x^*\hat{J}$, and here $\hat{J} : L^2(G) \to L^2(G)$ is just the map of pointwise conjugation. If G is a [SIN] group, then whenever V is an invariant neighbourhood of the identity, then $\xi_V = |V|^{-1/2}\chi_V$ will be an invariant unit vector in $L^2(G)$. Arguing as in [48, Section 3] we see that ω_ξ is a tracial state in $A(G)$. Let $\mathcal{U}$ be an ultrafilter refining the order filter on the set I of invariant neighbourhoods of the identity in G, and let Φ be the weak*-limit, taken in $VN(G)^*$, along $\mathcal{U}$, of the net (ω_{ξ_V}). Thus Φ is a trace on $VN(G)$. Define

$$\alpha : VN(G \times G) \to VN(G); \quad \alpha(x) = \lim_{\mathcal{U}} (\omega_{\xi_V} \otimes \iota)(W(x \otimes y)W^*),$$

where the limit is in the weak*-topology on $VN(G)$. For $\omega \in A(G)$,

$$\langle \alpha(x \otimes y), \omega \rangle = \lim_{\mathcal{U}} \langle (R(x)\omega_{\xi_V}) \star y, \omega \rangle = \lim_{\mathcal{U}} \langle \Delta(y)(R(x) \otimes 1), \omega_{\xi_V} \otimes \omega \rangle = \Phi\big((\omega \star y)R(x)\big).$$

Let (H, π, ξ_0) be the GNS construction for Φ. Hence

$$\langle \alpha(x \otimes y), \omega \rangle = \big(\pi(\omega \star y)\xi_0 \big| \pi(R(x^*))\xi_0\big).$$

We wish to find H is a more concrete way, for which we turn to the notion of an ultrapower of a Banach space, [21]. Let $\ell^\infty(L^2(G), I)$ be the Banach space of bounded families of vectors in $L^2(G)$ indexed by I. Define a degenerate inner-product on $\ell^\infty(L^2(G), I)$ by

$$\big((\xi_i)\big|(\eta_i)\big) = \lim_{i \to \mathcal{U}} (\xi_i | \eta_i).$$

The null-space is $N_\mathcal{U} = \{(\xi_i) : \lim_{i \to \mathcal{U}} \|\xi_i\| = 0\}$ and $\ell^\infty(L^2(G), I)/N_\mathcal{U}$ becomes a Hilbert space, denoted by $(L^2(G))_\mathcal{U}$. The equivalence class defined by (ξ_i) will be denoted by $[\xi_i]$. In particular, set $\xi_1 = [\xi_V]$. Any $T \in \mathcal{B}(L^2(G))$ acts on $(L^2(G))_\mathcal{U}$ by $T[\xi_i] = [T(\xi_i)]$. It is now easy to verify that the map

$$\pi(x)\xi_0 \mapsto [x(\xi_V)] = x\xi_1 \qquad (x \in VN(G))$$

is an isometry, and so extends to an isometric embedding $H \to (L^2(G))_\mathcal{U}$. We shall henceforth identify H with a closed subspace of $(L^2(G))_\mathcal{U}$.

Lemma 5.5. *For $x \in VN(G)$ and $\omega \in A(G)$, we have that $(\omega \star x)\xi_1 = (\omega \otimes \iota)(W^*)x\xi_1$. Hence H is an invariant subspace of $(L^2(G))_\mathcal{U}$ for the action of $C_0(G)$.*

Proof. A direct calculation easily establishes that

$$\lim_{V \to \{e\}} \|W(f \otimes \xi_V) - f \otimes \xi_V\| = 0 \qquad (f \in L^2(G)).$$

Let $g \in L^2(G)$, and let $(\overline{g} \otimes \iota) : L^2(G \times G) \to L^2(G)$ be the operator $\xi \otimes \eta \mapsto (\xi|g)\eta$. Then, for $\omega = \omega_{f,g} \in A(G)$,

$$\begin{aligned}
&\|(\omega \star x)\xi_1 - (\omega \otimes \iota)(W^*)x\xi_1\| \\
&\quad = \lim_{V \to \mathcal{U}} \|(\overline{g} \otimes \iota)W^*(1 \otimes x)W(f \otimes \xi_V) - (\overline{g} \otimes \iota)W^*(1 \otimes x)(f \otimes \xi_V)\| \\
&\quad = \lim_{V \to \mathcal{U}} \|(\overline{g} \otimes \iota)W^*(1 \otimes x)\big(W(f \otimes \xi_V) - f \otimes \xi_V\big)\| = 0,
\end{aligned}$$

as required. As $\{(\omega \otimes \iota)(W^*) : \omega \in A(G)\}$ is dense in $C_0(G)$, by continuity, it follows that $C_0(G)$, acting on $(L^2(G))_{\mathcal{U}}$, maps H to H. □

Let $\hat{\pi} : C_0(G) \to \mathcal{B}(H)$ the resulting $*$-homomorphism. For $\xi, \eta \in H$, it follows that $\omega_{\xi,\eta} \circ \hat{\pi} : C_0(G) \to \mathbb{C}$ is a functional, and so defines a measure in $M(G) = C_0(G)^*$. Then left convolution by the measure defines a member of $VN(G)$ (actually, of $M(C_r^*(G))$) which we shall denote by $\mu_{\xi,\eta}$. Notice that for $\mu \in M(G)$, the convolution operator so defined is $(\iota \otimes \mu)(W^*)$, which makes sense, as $W^* \in M(C_r^*(G) \otimes C_0(G))$. For the following result, notice that for $x \in VN(G)$, we have that $R(x^*)\xi_1 = [\overline{x\xi_V}]$ and so it follows that $\mu_{y\xi_1,R(x)^*\xi_1} = \mu_{x\xi_1,R(y)^*\xi_1}$.

Proposition 5.6. *We have that $\alpha(x \otimes y) = \mu_{y\xi_1,R(x)^*\xi_1}$. Consequently, for $z \in VN(G)$ with $\Delta(z) \in VN(G) \otimes VN(G)$, we have that z is in the norm closure of $M(G)$ inside $VN(G)$.*

Proof. For $\omega \in A(G)$, we have that

$$\begin{aligned}\langle \alpha(x \otimes y), \omega \rangle &= \big(\pi(\omega \star y)\xi_0 \big| \pi(R(x^*))\xi_0\big) = \big((\omega \otimes \iota)(W^*)y\xi_1 \big| R(x^*)\xi_1\big) \\ &= \langle (\omega \otimes \iota)(W^*), \omega_{y\xi_1,R(x^*)\xi_1} \circ \hat{\pi}\rangle = \langle \mu_{y\xi_1,R(x^*)\xi_1}, \omega\rangle.\end{aligned}$$

For $\epsilon > 0$, we can find $\tau \in VN(G) \odot VN(G)$ with $\|\Delta(z) - \tau\| < \epsilon$. Then

$$\|z - \alpha(\tau)\| = \|\alpha\Delta(z) - \alpha(\tau)\| < \epsilon.$$

We have just established that $\alpha(\tau) \in M(G)$ (inside $VN(G)$) and so the result follows. □

Of course, we would like to prove that such z are actually in $C_\delta^*(G)$. Let M denote the norm closure of $M(G)$ in $VN(G)$. Consider the closure of $\{\pi(\mu)\xi_0 : \mu \in M\}$ in H. We shall shortly see that this Hilbert space is isomorphic to $\ell^2(G)$. However, we have been unable to decide if this is all of H or not. Furthermore, just knowing that $\Delta(z) \in VN(G) \otimes VN(G)$ and that $z \in M$ does not tell us that we can approximate $\Delta(z)$ by an element of $M \odot M$. In the classical situation, when we compute the Bohr compactification of G (and not $\hat{G}$) then all our C*-algebras are commutative, and so they all have the approximation property, and so knowing, for example, that $\omega \star z \in M$ for all ω does tell us that $z \in M \otimes M$. In our setting, working with operator spaces, it seems very unlikely that M will have the required operator space version of the approximation property. We consequently impose a slightly stronger condition; see Theorem 5.3 below.

Define $\theta : \ell^2(G) \to H$ by $\delta_s \mapsto \lambda(s)\xi_1$. We note that

$$\big(\lambda(s)\xi_1 \big| \lambda(t)\xi_1\big) = \lim_{V \to \mathcal{U}} \langle \lambda(t^{-1}s), \omega_{\xi_V}\rangle = \delta_{t,s},$$

because if $t^{-1}s$ is not the identity, then $\langle \lambda(t^{-1}s), \omega_{\xi_V}\rangle$ will be zero for a sufficiently small neighbourhood V. It follows that θ is an isometry, and so $\theta\theta^*$ is the orthogonal projection of H into $\theta(\ell^2(G))$.

Lemma 5.7. *The action of $C_0(G)$ on H leaves $\ell^2(G)$ invariant, and so $\mu_{\theta(h),\eta} = \mu_{\theta(h),\theta\theta^*\eta} \in \ell^1(G)$ for all $h = (h_s) \in \ell^2(G)$ and $\eta \in H$.*

Proof. For $f \in C_0(G)$, we find that

$$\begin{aligned}\lim_{V \to \mathcal{U}} \|f\lambda(s)\xi_V - f(s)\lambda(s)\xi_V\|^2 &= \lim_{V \to \mathcal{U}} \int_G |f(t)\xi_V(s^{-1}t) - f(s)\xi_V(s^{-1}t)|^2 \, dt \\ &= \lim_{V \to \mathcal{U}} |V|^{-1} \int_G |f(t) - f(s)|^2 \chi_V(s^{-1}t) \, dt.\end{aligned}$$

Now, $\chi_V(s^{-1}t) = 0$ unless $t \in sV$, a small neighbourhood of s. As f is continuous, $|f(t)-f(s)|^2$ will be, on average, small on the set sV, and hence the limit is zero. It follows that $f\theta(\delta_s) = f(s)\theta(\delta_s)$, and so

$$\langle f, \mu_{\theta(\delta_s),\eta}\rangle = f(s)\big(\theta(\delta_s)\big|\eta\big) = f(s)\big(\theta(\delta_s)\big|\theta\theta^*\eta\big) = \langle f, \mu_{\theta(\delta_s),\theta\theta^*\eta}\rangle,$$

as required. □

Lemma 5.8. *Let $\mu \in M(G)$, and treat μ as an operator in $VN(G)$. Let $a \in \ell^1(G) \subseteq \ell^2(G)$ be the atomic part of μ. Then $\mu\xi_1 = \theta(a)$.*

Proof. Let $\mu, \nu \in M(G)$ and note that $M(G)$ is a Banach $*$-algebra (the $*$-operation is $\langle \mu^*, a\rangle = \int \overline{a(s^{-1})} d\mu(s)$ for $a \in C_0(G)$) with the natural map $M(G) \to VN(G)$ a $*$-homomorphism. Then

$$(\mu\xi_1|\nu\xi_1) = \varphi(\nu^*\mu) = \lim_{V\to\mathcal{U}} \big(\nu^*\mu\xi_V\big|\xi_V\big) = \lim_{V\to\mathcal{U}} \langle \nu^*\mu, (\omega_V \otimes \iota)(W^*)\rangle.$$

If we let $a_V = (\omega_V \otimes \iota)(W^*) \in C_0(G)$, then it's easy to see that $a_V(e) = 1$ for all V (with e the unit of G) and that for any open set U containing e, eventually the support of a_V is contained in U. As $\nu^*\mu$ is a regular measure, it follows that

$$\lim_{V\to\mathcal{U}} \langle \nu^*\mu, (\omega_V \otimes \iota)(W^*)\rangle = (\nu^*\mu)(\{e\}),$$

the measure of the singleton $\{e\}$. However, for any Borel set E we have that (see [22, Theorem 19.11])

$$(\nu^*\mu)(E) = \int_G \nu^*(Es^{-1})\, d\mu(s).$$

Now, for any $s \in G$, we have that $\nu^*(\{s^{-1}\}) = \overline{\nu(\{s\})}$ and there is hence a countable (or possibly finite) subset $F \subseteq G$ with $\nu^*(\{s^{-1}\}) = 0$ if $s \notin F$. So the function $s \mapsto \nu^*(\{s^{-1}\})$ is countably supported, and hence

$$(\nu^*\mu)(\{e\}) = \int_G \nu^*(\{s^{-1}\})\, d\mu(s) = \sum_{s\in G} \overline{\nu(\{s\})}\mu(\{s\}).$$

If $a, b \in \ell^1(G)$ are the atomic parts of μ and ν respectively, then it follows that $(\nu^*\mu)(\{e\}) = (b^*a)(\{e\})$ and hence $(\mu\xi_1|\nu\xi_1) = (a\xi_1|b\xi_1)$.

It hence follows that

$$\|\mu\xi_1 - a\xi_1\|^2 = (\mu\xi_1|\mu\xi_1) - (\mu\xi_1|a\xi_1) - (a\xi_1|\mu\xi_1) + (a\xi_1|a\xi_1) = 0.$$

The proof is then complete by observing that if $\iota : \ell^1(G) \to \ell^2(G)$ is the formal identity, then $\theta\iota(a) = a\xi_1$. □

Theorem 5.9 (Theorem 5.3)**.** *Let G be a [SIN] group, and let $x_0 \in VN(G)$ be such that $\Delta^2(x_0) \in VN(G) \otimes VN(G) \otimes VN(G)$. Then $x_0 \in C^*_\delta(G)$.*

Proof. Define $\overline{\alpha} : VN(G)\overline{\otimes}VN(G)\overline{\otimes}VN(G) \to VN(G)\overline{\otimes}VN(G)$ by

$$\langle \overline{\alpha}(x), \omega \otimes \tau\rangle = \langle \alpha((\iota \otimes \iota \otimes \tau)(x)), \omega\rangle.$$

We should justify why this makes sense. Notice that for $f, g \in L^2(G)$,

$$\big(\alpha(x)f\big|g\big) = \big(x[W^*(\xi_V \otimes f)]\big|[W^*(\xi_V \otimes g)]\big),$$

where $[W^*(\xi_V \otimes f)]$ is an element of $(L^2(G \times G))_{\mathcal{U}}$. It's now clear that α is completely bounded, and now standard operator space techniques show that $\overline{\alpha}$ exists, and is completely bounded with $\|\overline{\alpha}\|_{cb} \leq \|\alpha\|_{cb} = 1$. For $x \in VN(G)$ and $\omega, \tau \in A(G)$,

$$\langle \overline{\alpha}\Delta^2(x), \omega \otimes \tau \rangle = \langle \alpha\big((\iota \otimes \iota \otimes \tau)\Delta^2(x)\big), \omega \rangle = \langle \alpha\Delta\big((\iota \otimes \tau)\Delta(x)\big), \omega \rangle = \langle \Delta(x), \omega \otimes \tau \rangle.$$

Hence $\overline{\alpha}\Delta^2(x) = \Delta(x)$.

For $\epsilon > 0$, we can find $u = \sum_{i=1}^n a_i \otimes b_i \otimes c_i \in VN(G) \odot VN(G) \odot VN(G)$ with $\|\Delta^2(x_0) - u\| < \epsilon$. Then

$$\Big\| x_0 - \sum_i \alpha\big(\alpha(a_i \otimes b_i) \otimes c_i\big) \Big\| = \big\| x_0 - \alpha\overline{\alpha}(u) \big\| < \epsilon.$$

However, for each i there is a measure μ_i with $\alpha(a_i \otimes b_i) = \mu_i$. Hence, by Proposition 5.6,

$$\sum_i \alpha\big(\alpha(a_i \otimes b_i) \otimes c_i\big) = \sum_i \alpha(\mu_i \otimes c_i) = \sum_i \mu_{\mu_i \xi_1, R(c_i)^* \xi_1}.$$

As $\mu_i \xi_1 \in \theta(\ell^2(G))$, by Lemma 5.7, it follows that the sum defines a member of $\ell^1(G)$. So x_0 can be norm approximated by elements of $\ell^1(G)$, that is, $x_0 \in C^*_\delta(G)$. □

We remark (without proof) that if $\Delta^2(x) \in VN(G) \otimes VN(G) \otimes VN(G)$ then using that $\Delta_* : A(G)\widehat{\otimes}A(G) \to A(G)$ is a complete surjection, one can show that $A(G) \to VN(G); \omega \mapsto \omega \star x$ is completely compact. It follows, as in [42], that if G is amenable or connected (so $VN(G)$ is injective) then $x \in \mathrm{CAP}(A(G))$. Conversely, if $\Delta(x) \in VN(G) \otimes VN(G)$ then $\Delta^2(x) \in VN(G \times G) \otimes VN(G) \cap VN(G) \otimes VN(G \times G)$, but we don't see why $\Delta^2(x)$ need be in $VN(G) \otimes VN(G) \otimes VN(G)$. Obviously $x \in C^*_\delta(G)$ does imply this, however.

6 Further examples

In this section, we study various examples, and present some counter-examples to conjectures in [43].

6.1 Commutative case

There is little to say here– the categorical construction obviously agrees with the usual *strongly almost periodic* or *Bohr compactification*, [7, 23, 24]. Furthermore, in the commutative case, there is no distinction between the reduced and universal case.

6.2 Reduced quantum groups

In [43, Question 2], Sołtan asked, in particular, if $\mathbb{AP}(C_0(\mathbb{G}))$ is always a reduced compact quantum group, or in our language, if $\mathbb{AP}(C_0(\mathbb{G})) \to C(\mathbb{G}^{\mathrm{SAP}})$ is an isomorphism. In this section, we shall show that the answer is "no", even if $\mathbb{G}$ is cocommutative.

As in the previous section, let $\mathbb{G} = \hat{G}$ for a locally compact group G. Then the compactification of $\mathbb{G}$ is $\widehat{G_d}$, so $C(\mathbb{G}^{\mathrm{SAP}}) = C^*_r(G_d)$ and $C^u(\mathbb{G}^{\mathrm{SAP}}) = C^*(G_d)$. We follow the notation of, in particular, [6], and write again $C^*_\delta(G)$ for the span of the translation operators $\{\lambda(s) : s \in G\}$ in $M(C_0(\mathbb{G})) = M(C^*_r(G))$. Following [6], consider the following surjective Hopf $*$-homomorphisms:

$$C^u(\mathbb{G}^{\mathrm{SAP}}) = C^*(G_d) \xrightarrow{\Psi} \mathbb{AP}(C^u_0(\mathbb{G})) \xrightarrow{\alpha} \mathbb{AP}(C_0(\mathbb{G})) = C^*_\delta(G) \xrightarrow{\Phi} C(\mathbb{G}^{\mathrm{SAP}}) = C^*_r(G_d).$$

Here we use the notation of [6], except that our map α is denoted by Λ there (which obviously clashes with other notation in this paper). Then [6] proves the following (some of these results also follow from work in [1] and [16]):

- α is an isomorphism if and only if G is amenable;
- $\Phi \circ \alpha$ is an isomorphism if and only if G_d is amenable;
- Φ is an isomorphism if and only if G contains an open subgroup H with H_d amenable;
- if G is a connected Lie group, then Ψ is an isomorphism if and only if G is solvable, if and only if Φ is an isomorphism. Recall that in this case, G is solvable if and only if G_d is amenable, [38, Theorem 3.9].

Example 6.1. In particular, we see that the compact quantum group $\mathbb{AP}(C_0(\mathbb{G}))$ is reduced if and only if Φ is an isomorphism. Setting $\mathbb{G}$ to be the dual of $SU(2)$ or $SO(3)$, we see that $\mathbb{G}$ is a discrete, cocommutative quantum group, and Φ is not an isomorphism, as G is connected, but G_d is not amenable (as it contains a free group, [38, Proposition 3.2]).

Example 6.2. Let G be an amenable, connected Lie group with G_d non-amenable (again, $G = SU(2)$ or $SO(3)$ works), and set $\mathbb{G} = \hat{G}$. Then $\mathbb{AP}(C_0^u(\mathbb{G})) = \mathbb{AP}(C_0(\mathbb{G}))$, but these are not equal to either $C_0^u(\mathbb{G}^{\mathrm{SAP}})$ nor to $C_0(\mathbb{G}^{\mathrm{SAP}})$. Such compact quantum groups, lying strictly between their universal and reduced versions, were studied in [31, Section 8], so this example gives a whole family of further "exotic" compact quantum group norms. Furthermore, as again $\mathbb{G}$ is a discrete quantum group, this answers in the negative a conjecture made after [43, Question 1], as $\mathbb{AP}(C_0^u(\mathbb{G}))$ is not universal.

We finish this section by observing that we can prove something like analogues for some of the above facts for general quantum groups.

There are many equivalent definitions of what it means for a general locally compact quantum group $\mathbb{G}$ to be coamenable. We shall choose the definition that $\mathbb{G}$ is *coamenable* if the counit is bounded on $C_0(\mathbb{G})$, see [4, Section 3]. That is, there is a state $\epsilon \in C_0(\mathbb{G})^*$ with $(\iota \otimes \epsilon)\Delta = \iota$. Then ϵ is unique, and $(\epsilon \otimes \iota)\Delta = \iota$. For a locally compact group G, always G is coamenable, while $\hat{G}$ is coamenable if and only if G is amenable.

Proposition 6.3. *Let $\mathbb{G}$ be a locally compact quantum group and consider the maps $\alpha : \mathbb{AP}(C_0^u(\mathbb{G})) \to \mathbb{AP}(C_0(\mathbb{G}))$ and $\Phi : \mathbb{AP}(C_0(\mathbb{G})) \to C(\mathbb{G}^{\mathrm{SAP}})$. Then:*

1. *$\Phi \circ \alpha : \mathbb{AP}(C_0^u(\mathbb{G})) \to C(\mathbb{G}^{\mathrm{SAP}})$ is injective (and hence an isomorphism) if and only if $\mathbb{G}^{\mathrm{SAP}}$ is coamenable.*
2. *Suppose that $\mathbb{G}$ is coamenable. Then the natural map $\Phi : \mathbb{AP}(C_0(\mathbb{G})) \to C(\mathbb{G}^{\mathrm{SAP}})$ is injective (that is, $\mathbb{AP}(C_0(\mathbb{G}))$ is reduced) if and only if $\mathbb{G}^{\mathrm{SAP}}$ is coamenable.*

Proof. For (1) we note that $C_0^u(\mathbb{G})$ always admits a bounded counit ϵ_u, see [27, Section 4]. If $\Phi \circ \alpha$ is injective then it's an isomorphism (as the dense Hopf $*$-algebras agree, see discussion around Definition 3.10). Thus the restriction of ϵ_u to $\mathbb{AP}(C_0^u(\mathbb{G}))$ induces a bounded counit on $C(\mathbb{G}^{\mathrm{SAP}})$ and so $\mathbb{G}^{\mathrm{SAP}}$ is coamenable. Conversely, if $\mathbb{G}^{\mathrm{SAP}}$ is coamenable then $C^u(\mathbb{G}^{\mathrm{SAP}}) = C(\mathbb{G}^{\mathrm{SAP}})$ (see [5, Theorem 2.2]) and so as the canonical surjection $C^u(\mathbb{G}^{\mathrm{SAP}}) \to C(\mathbb{G}^{\mathrm{SAP}})$ factors through $\Phi \circ \alpha$, it follows that $\Phi \circ \alpha$ is injective.

For (2), let $\epsilon \in C_0(\mathbb{G})^*$ be the counit, which exists as $\mathbb{G}$ is coamenable. If Φ is injective, then it is an isomorphism, and the restriction of ϵ to $\mathbb{AP}(C_0(G)) \cong C(\mathbb{G}^{\mathrm{SAP}})$ defines a bounded counit on $C(\mathbb{G}^{\mathrm{SAP}})$, showing that $\mathbb{G}^{\mathrm{SAP}}$ is coamenable. Conversely, if $\mathbb{G}^{\mathrm{SAP}}$ is coamenable then by [5, Theorem 2.2] the Haar state on $\mathbb{AP}(C_0(G))$ is faithful and so Φ is injective. $\square$

6.3 Compact quantum groups

The whole theory is designed to ensure that if $\mathbb{G}$ is a compact quantum group, then it is its own compactification (compare [43, Section 4.3]). Of interest here are links with Section 4. The following is an improvement upon Theorem 4.14, in that we make no assumption about the antipode. The proof is very similar to [56, Theorem 2.6(2)], where it is shown that, when $\mathbb{G}$ is compact, if $a \in \mathcal{P}^\infty(\mathbb{G}) \cap C_0(\mathbb{G})$, then $a \in \mathcal{AP}(C_0(\mathbb{G}))$. We give the details, because the argument is not long, and makes an interesting link with Theorem 4.14. Our proof avoids use of $L^2(\mathbb{G})$, and so *maybe* holds promise of extension to the non-compact case.

Theorem 6.4. *Let $\mathbb{G}$ be compact. Then $\mathcal{P}^\infty(\mathbb{G}) = \mathcal{AP}(C_0(\mathbb{G}))$.*

Proof. It suffices to show that $x \in \mathcal{P}^\infty(\mathbb{G})$ is in $\mathcal{AP}(C_0(\mathbb{G}))$. Let φ be the (normal) Haar state on $L^\infty(\mathbb{G})$. By [30, Section 1], compare also [56, Theorem 2.6(4)], we know that for $a, b \in L^\infty(\mathbb{G})$,

$$(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b) \in D(S), \quad S\big((\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))\big) = (\iota \otimes \varphi)((1 \otimes a^*)\Delta(b)).$$

For $a \in L^\infty(\mathbb{G})$, let $\omega_a \in L^1(\mathbb{G})$ be the functional $\langle b, \omega_a \rangle = \varphi(a^* b)$ for $b \in L^\infty(\mathbb{G})$. As φ is a KMS state, such functionals are dense in $L^1(\mathbb{G})$.

That $x \in \mathcal{P}^\infty(\mathbb{G})$ means that $\Delta(x) = \sum_{i=1}^n x_i \otimes y_i$ with $\{x_i\}$ and $\{y_i\}$ linearly independent sets. Arguing as in the proof of Theorem 4.14, we can find $(a_i) \subseteq L^\infty(\mathbb{G})$ such that $\langle y_j, \omega_{a_i} \rangle = \delta_{ij}$. Thus for each i,

$$S\big((\iota \otimes \varphi)(\Delta(a_i^*)(1 \otimes x))\big) = (\iota \otimes \varphi)((1 \otimes a_i^*)\Delta(x)) = \sum_j x_j \langle y_j, \omega_{a_i} \rangle = x_i.$$

So $x_i \in D(S^{-1}) = D(S)^*$. Applying the same argument to x^* shows that $x_i \in D(S)$. Applying the same argument to $\mathbb{G}^{\mathrm{op}}$ shows that $y_i \in D(S) \cap D(S)^*$ for all i.

A close examination of the proof of Theorem 4.14 shows that knowing that $x_i, y_i \in D(S) \cap D(S)^*$ for all i is enough for the proof to work, and so $x \in \mathcal{AP}(C_0(\mathbb{G}))$, as required. □

We remark that Woronowicz asked in [56] if it was essential to focus on *reduced* compact quantum groups for this result hold. This was answered affirmatively in [31, Remark 9.6]; in our language, a compact quantum group (A, Δ) is constructed, and an element $a \in A$ is found, such that $\Delta(a)$ is a finite-rank tensor in $A \odot A$, but $a \notin \mathcal{AP}(A)$.

6.4 Discrete quantum groups

Discrete quantum groups were extensively studied in [43]. An important tool is the *canonical Kac quotient* of a compact quantum group, an idea attributed to Vaes. Given a compact quantum group (A, Δ), let I be the closed ideal formed of all $a \in A$ such that $\tau(a^* a) = 0$ for all traces τ. If A admits no traces, set $I = A$. Let $A_{\mathrm{KAC}} = A/I$ with $\pi : A \to A_{\mathrm{KAC}}$ the quotient map. Then

$$\Delta_{\mathrm{KAC}}(a + I) = (\pi \otimes \pi)\Delta(a) \qquad (a \in A),$$

is well-defined, and $(A_{\mathrm{KAC}}, \Delta_{\mathrm{KAC}})$ becomes a compact quantum group. It turns out that the Haar state is tracial, and so $(A, \Delta_{\mathrm{KAC}})$ is a Kac algebra. In particular, the dual of $(A, \Delta_{\mathrm{KAC}})$ is a unimodular discrete quantum group, and hence has a bounded antipode. All this is explained in [43, Appendix A].

Let us now adapt the argument given in [43, Section 4.3] and single out a key idea. Let $\mathbb{G}$ be discrete and set $(A, \Delta) = (C^u(\hat{\mathbb{G}}), \Delta^u_{\hat{\mathbb{G}}})$. Then $(A_{\mathrm{KAC}}, \Delta_{\mathrm{KAC}})$ is a compact Kac algebra, and so also its universal form, say $(C^u(\hat{\mathbb{H}}), \Delta^u_{\hat{\mathbb{H}}})$ is Kac. Let $\pi_u : C^u(\hat{\mathbb{G}}) \to C^u(\hat{\mathbb{H}})$ be the unique lift of $\pi : A \to A_{\mathrm{KAC}}$, and let $\hat{\pi} : C_0(\mathbb{H}) \to C_0(\mathbb{G})$ be the dual (recall that $\mathbb{G}$ and $\mathbb{H}$ are discrete, and so $C_0^u(\mathbb{H}) = C_0(\mathbb{H})$ and so forth).

Proposition 6.5. *For all n, the map $V \mapsto (\hat{\pi} \otimes \iota)(V)$ gives a surjection from the set of n-dimensional unitary corepresentations $V \in M(C_0(\mathbb{H})) \otimes \mathbb{M}_n$ to the set of n-dimensional unitary corepresentations of $C_0(\mathbb{G})$.*

Proof. As $\hat{\pi}$ is a Hopf $*$-homomorphism, we need only prove surjectivity of the map; namely, that if $U \in M(C_0(\mathbb{G})) \otimes \mathbb{M}_n$ is a unitary corepresentation, then $U = (\hat{\pi} \otimes \iota)(V)$ for some suitable V. Recall again the work of Kustermans in [27]. There is a unique $*$-homomorphism $\phi : A \to \mathbb{M}_n$ with $U = (\iota \otimes \phi)(\hat{\mathcal{V}}_{\mathbb{G}})$. Furthermore, as $\mathbb{G}$ and $\mathbb{H}$ are discrete,

$$\mathcal{U}_{\mathbb{G}} = \hat{\mathcal{V}}_{\mathbb{G}}, \mathcal{U}_{\mathbb{H}} = \hat{\mathcal{V}}_{\mathbb{H}} \implies (\iota \otimes \pi_u)(\hat{\mathcal{V}}_{\mathbb{G}}) = (\hat{\pi} \otimes \iota)(\hat{\mathcal{V}}_{\mathbb{H}}).$$

As $\mathbb{M}_n$ has a faithful trace, it is easy to see that there is a unique $*$-homomorphism $\phi_0 : A_{\mathrm{KAC}} \to \mathbb{M}_n$ with $\phi_0 \circ \pi = \phi$. Recall the reducing morphism $\Lambda^u_{A_{\mathrm{KAC}}} : C^u(\hat{\mathbb{H}}) \to A_{\mathrm{KAC}}$, and set $V = (\iota \otimes \phi_0 \circ \Lambda^u_{A_{\mathrm{KAC}}})(\hat{\mathcal{V}}_{\mathbb{H}})$, an n-dimensional unitary corepresentation of $C_0(\mathbb{H})$. Then

$$(\hat{\pi} \otimes \iota)(V) = (\iota \otimes \phi_0 \circ \Lambda^u_{A_{\mathrm{KAC}}} \circ \pi_u)(\hat{\mathcal{V}}_{\mathbb{G}}) = (\iota \otimes \phi_0 \circ \pi)(\hat{\mathcal{V}}_{\mathbb{G}}) = U,$$

as required. □

Now, Corollary 4.17 shows that finite-dimensional unitary corepresentations of $\mathbb{H}$ are automatically admissible (a result not available to Sołtan) and so we get the following (which Sołtan was able to prove by other means, see [43, Theorem 4.5]). We will revisit this result below.

Corollary 6.6. *Any finite-dimensional unitary corepresentation of a discrete quantum group $\mathbb{G}$ is admissible.*

Proof. Let U be a finite-dimension unitary corepresentation of $C_0(\mathbb{G})$. Then $U = (\hat{\pi} \otimes \iota)(V)$ for an (automatically) admissible unitary corepresentation V of $C_0(\mathbb{H})$. Then, as in the proof of Proposition 3.9, $\overline{V}$ is similar to a unitary corepresentation, say X. A simple calculation then shows that $\overline{U} = (\hat{\pi} \otimes \iota)(\overline{V})$ is similar to $(\hat{\pi} \otimes \iota)(X)$ and hence admissible. □

7 Open questions

The most interesting open question seems to be:

Conjecture 7.1. For any $\mathbb{G}$, we have that $\mathcal{P}^\infty(\mathbb{G}) = \mathcal{AP}(\mathbb{G})$.

One obvious attack is suggested by Theorem 4.14: show that if $x \in \mathcal{P}^\infty(\mathbb{G})$ then automatically $x \in D(S) \cap D(S)^*$. Woronowicz's argument, Theorem 6.4, shows that this is true for compact $\mathbb{G}$.

In the extreme case of one-dimensional corepresentations, the answer is also affirmative. To be precise, if $x \in L^\infty(\mathbb{G})$ is a corepresentation, meaning that $\Delta(x) = x \otimes x$, then automatically x is unitary (and so $x \in D(S) \cap D(S)^*$). Two independent proofs are given in [13, Theorem 3.2] and [26, Theorem 3.9], but both proofs use, for example, that also x^*x is a character, and there seems little hope of extending these sorts of arguments to more general periodic elements.

A weaker conjecture is the following:

Conjecture 7.2. For any $\mathbb{G}$, the finite-dimensional unitary corepresentations of $C_0(\mathbb{G})$ are admissible.

This is true for compact quantum groups from Woronowicz's work (see [56, Proposition 6.2], which is the key to showing that the matrix elements of unitary corepresentations form a Hopf $*$-algebra, in the compact case). It is true for Kac algebras by Corollary 4.17, and is true for discrete quantum groups by Corollary 6.6. Recall that the final result is proved by using the "canonical

Kac quotient" of the dual: any finite-dimensional $*$-representation of a compact quantum group factors through a (compact) Kac algebra; but this technique is something special to the compact case, and fails for discrete quantum groups, for example. However, we wonder if some slightly different technique could be used to prove the conjecture? We note that in all computations of the quantum Bohr compactification, one computes the finite-dimensional unitary corepresentations of $C_0(\mathbb{G})$ via computing the finite-dimensional $*$-representations of $C_0^u(\hat{\mathbb{G}})$, and then in each special case, it turns out that these corepresentations are always admissible.

In the classical situation, consider the link between finite-dimensional unitary representations π of G in $\mathbb{M}_n$, and group homomorphisms from G to compact groups. Trivially, any such π induces a group homomorphism $G \to U(n)$; and the Peter-Weyl theory tells us that to understand homomorphisms $G \to K$ for compact K, it is enough (in some sense) to know the finite-dimensional unitary representations of G. This second point was of course generalised by Sołtan in [43]. However, the first point has links to Conjecture 7.2. The following is easy to show, as every finite-dimensional corepresentation of a compact quantum group is admissible.

Proposition 7.3. *Conjecture 7.2 holds for $\mathbb{G}$ if and only if every finite-dimensional unitary corepresentation U of $\mathbb{G}$ factors through a compact quantum group.*

We remark that the quantum group analogues of the unitary groups are the "universal" quantum groups, in the sense of van Daele and Wang, [54, 52]. If Conjecture 7.2 is false, then there are finite-dimensional unitary corepresentations of $\mathbb{G}$ which have nothing to do with compact quantum groups: a very strange situation!

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