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Broadband energy harvesting from parametric vibrations of a class of nonlinear Mathieu systems

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The nonlinear dynamics of the Mathieu equation with the inclusion of a cubic stiffness component is considered for broadband vibration energy harvesting. The results of numerical integration are compared with the corresponding solution of a regular Duffing oscillator which is widely used to model nonlinear energy harvesting. The use of Duffing oscillators has shown direct correspondence between the effective frequency range of the associated hysteretic phenomenon and the value of the nonlinearity coefficient. A broadband energy harvester requires strong nonlinearity, especially for high frequencies of interest. This letter demonstrates that the effectiveness of parametrically excited systems is not constrained by the same requirement. Based on this, it is suggested that parametrically excited systems can be a robust means of broadband vibration harvesting.
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Recent advances in the electronics industry have resulted in the manufacture of compact devices, operating with minimal energy requirements, typically of the order of μW to mW. Vibration energy is viewed as a potential source of renewable energy, which can power small-scale electronic systems.[1] The underlying assumption is that ambient vibrations can be harvested from moving bodies and coupled to electrodynamic elements, such as piezoelectric patches.[2] The primary principle is that the vibration energy which is converted into electrical power can potentially power operating electronic devices, locally or remotely.

The challenge in effective energy harvesting has been its dependence upon the ratio of excitation frequency to the natural frequency of a harvester. As long as this ratio remains close to unity, the resonant conditions result in optimal device performance.[3,4] However, ambient vibrations are subject to variations in the response frequency, affecting the aforementioned ratio and deviating from the optimal conditions. In recent times, a number of studies have addressed this problem, proposing the intentional use of nonlinearities in the system stiffness.[5–7] Nonlinear oscillators are well-known in the hysteretic response, which enables multiple stable solutions to co-exist over a wide range of frequencies. The physical properties of nonlinear dynamical systems such as bi-stability or stochastic resonance have also been suggested in order to improve the performance of vibration energy harvesters.[5,6] In particular, systems with cubic stiffness elements, also known as Duffing oscillators, attain a high or low energy solution in the frequency range of their backbone curve. This characteristic amplitude-frequency response curve may be deliberately used to increase the frequency region for sustained effective energy harvesting. In this approach, the degree of success depends on the relationship between the amplitude of excitation and the damping coefficient,[4] as well as the magnitude of the nonlinear force coefficient. A typical mono-stable energy harvester, governed by a Duffing-like equation, is usually designed in such a way that the bent response curve would cover the frequency range of interest. For given input energy and damping, one would rely on the magnitude of the nonlinear coefficient to design a harvester which would cover a sufficiently wide frequency range.

It is well established that higher values of the nonlinear stiffness coefficient enable a broader frequency range over which the response curve bends.[10] Nevertheless, the practical implementation of this strategy is limited by the extent to which the nonlinear coefficient can be increased independent of linear stiffness elements, as in applied nonlinear magnetic force for instance. The problem becomes more intractable at higher frequencies, where stiffer systems would be required. For example, the weakly nonlinear response bandwidth of a mono-stable Duffing-like harvester does not significantly differ from that of a linear system, and thus, any desired improvement is lost.

This letter presents an alternative approach for utilizing nonlinear vibrations in energy harvesting applications. The aim is to disassociate the frequency band from the magnitude of the nonlinear stiffness coefficient, thus allowing for broadband energy harvesting in weakly nonlinear systems as well. An important class of dynamical systems is represented by the Mathieu equation. Parametric excitation and the associated resonant zones which lead to unbound responses characterize the dynamics of such systems. These zones occur at a plethora of frequency ratios $2/n$, with $n = 1, 2, \ldots$. When a nonlinear spring restores the motion of the oscillator, the unstable motion turns to the periodic orbit,[10] which may serve vibration energy harvesting purposes.

Parametrically excited systems have been discussed in the literature for energy harvesting applications. A cantilevered piezoelectric beam with a tip mass has been considered

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under axial excitations, a setting which can potentially lead to parametric instability.[9,10] Multiple scale analyses have demonstrated the influence of the critical excitation amplitude, giving rise to stable oscillations. Daqaq and Bode[11] exploited parametric resonance in order to channel the energy away from the superharmonic components of the excitation to their fundamental resonant response. In this manner, parametric excitation is used to amplify the response of a cantilever beam to regular external excitations, yielding enhanced harvesting performance. Yet, the effect of parametric excitation on the output power bandwidth has not been considered. Zaghari *et al.*[12] considered a cantilever beam with an additional nonlinear magnetic restoring force. They investigated, analytically and experimentally, the transition curves, leading to stable oscillations. Furthermore, harvesters employing pendula have also been considered for their rotational response to parametric excitation.[13–15] Jia *et al.*[14] developed a novel harvesting device, but their experiments only focused on the excitation amplitude and the dependence of the harvester's performance on the amplitude threshold associated with parametric excitation. The analyses thus far have not addressed the qualitative differences between parametrically and externally excited nonlinear systems over the resonant frequency range. Herein, it is demonstrated that the former offers a wider useful frequency range for weakly nonlinear systems, thus potentially presenting a route for curtailing the requirement for the presence of strong nonlinearities.

Figure 1 shows a generic electromagnetic energy harvester, where the magnetic mass undergoes parametric and direct force excitations in the proximity of a coil. The dynamics of this system can be described as ($'$ denotes the derivative with respect to time)

$$x'' + 2\gamma x' + (\delta + 2r\alpha_e \cos\nu\tau)x + \beta x^3 + \frac{\Theta}{m} I = 2(1-r)\alpha_e \cos\nu\tau, \tag{1}$$

$$LI' + RI - \Theta x' = 0, \tag{2}$$

where x is the oscillator's displacement, γ is a normalised damping coefficient, δ denotes the natural frequency, a_e is the excitation amplitude, ν is the excitation frequency, τ denotes the dimensionless time, β is the nonlinear coefficient, Θ is the coupling coefficient, m is the oscillator's mass, L is the coil inductance, I is the current, and R is the coil's resistance. Parameter r is introduced so that Eq. (1)

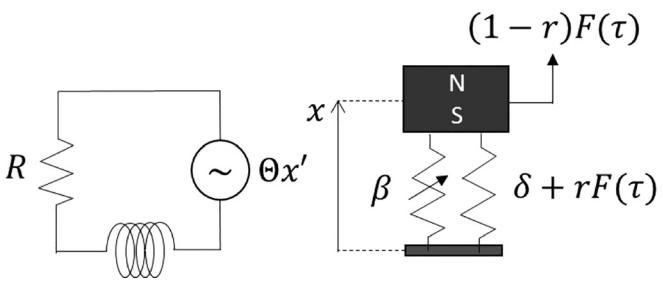

FIG. 1. Sketch of the electromagnetic harvester model where harmonic forcing applies, $F(\tau) = 2\alpha_e \cos\nu\tau$. The stiffness coefficient undergoes time-dependent variations for $r = 1$, leading to parametrically excited dynamics; $r = 0$ leads to an externally excited system with a constant stiffness coefficient.

devolves into a directly forced Duffing system when $r = 0$ and to a nonlinear Mathieu system when $r = 1$. After non-dimensionalisation of the current, $j = LI/\Theta$, and after the introduction of $\rho = R/L$ and $\theta = \Theta^2/mL$, Eqs. (1) and (2) become

$$x'' + 2\gamma x' + (\delta + 2r\alpha_e \cos\nu\tau)x + \beta x^3 + \theta j = 2(1-r)\alpha_e \cos\nu\tau, \tag{3}$$

$$j' + \rho j - x' = 0. \tag{4}$$

One can use a first order multiple scale approximation in order to study the dynamics of Eq. (3). Assuming $\tau = T_0 + \varepsilon T_1$, $\gamma = \varepsilon\gamma$, $\alpha_e = \varepsilon\alpha_e$, $\beta = \varepsilon\beta$, $\theta = \varepsilon\theta$, and $\varepsilon \ll 1$, it follows that

$$\begin{aligned} x &= x_0 + \varepsilon x_1, \\ j &= j_0 + \varepsilon j_1. \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eqs. (3) and (4), defining $\delta = \omega^2$, and collecting similar powers of ε result in

$$\begin{aligned} \varepsilon^0: \quad & D_0^2 x_0 + \omega^2 x_0 = 0, \\ & Dj_0 + \rho j_0 = D_0 x_0, \end{aligned} \tag{6}$$

$$\begin{aligned} \varepsilon^1: \quad & D_0^2 x_1 + \omega^2 x_1 = -2D_0 D_1 x_0 - 2\gamma D_0 x_0 \\ & \qquad - 2\alpha_e x_0 \cos\nu T_0 - \beta x_0^3 - \theta j_0, \\ & D_0 j_1 + \rho j_1 = -D_1 j_0 + D_1 x_0 + D_0 x_1, \end{aligned} \tag{7}$$

where the operator D_m^k denotes the k-th partial derivative with respect to T_m. The necessary analytical treatment of Eqs. (6) and (7) is omitted due to its triviality and for the sake of brevity. Thus, only the useful worked expressions are provided. The reader can refer to Ref. 8. Using Eq. (6), it can be seen that

$$x_0 = \frac{a}{2} e^{ib} e^{i\omega T_0} + c.c., \tag{8}$$

where a denotes the amplitude and b a phase difference. Substituting Eq. (8) into Eq. (6), solving for j_0 and substituting into Eq. (7), setting $\nu = 2\omega + \varepsilon\sigma$, and finally setting the secular terms (i.e., proportional to $e^{i\omega T_0}$) to be zero result in

$$\begin{aligned} a' &= \frac{1}{2}\left(2\gamma + \frac{\theta\rho}{\rho^2+\omega^2}\right)\alpha - \frac{\alpha_e a}{2\omega}\sin\varphi, \\ \alpha\varphi' &= \alpha\sigma - \frac{\alpha_e a}{\omega}\cos\varphi - \frac{3}{4\omega}\beta\alpha^3 - \frac{\theta\omega\alpha}{\rho^2+\omega^2}, \end{aligned} \tag{9}$$

where $\varphi = \sigma T_1 - 2b$. Seeking a steady-state non-trivial response, $a' = \varphi' = 0$, the set of Eq. (9) are squared and added to obtain the amplitude-frequency expression as

$$a^2 = \frac{4}{3\beta}\left[\sigma\omega - \frac{\theta\omega^2}{\rho^2+\omega^2} \pm \sqrt{\alpha_e^2 - \omega^2\left(2\gamma + \frac{\theta\rho}{\rho^2+\omega^2}\right)^2}\right]. \tag{10}$$

Considering $x \approx x_0$, Eq. (10) gives the response amplitude for the first approximation. Furthermore, taking the covariant (total) derivative of x with respect to τ yields

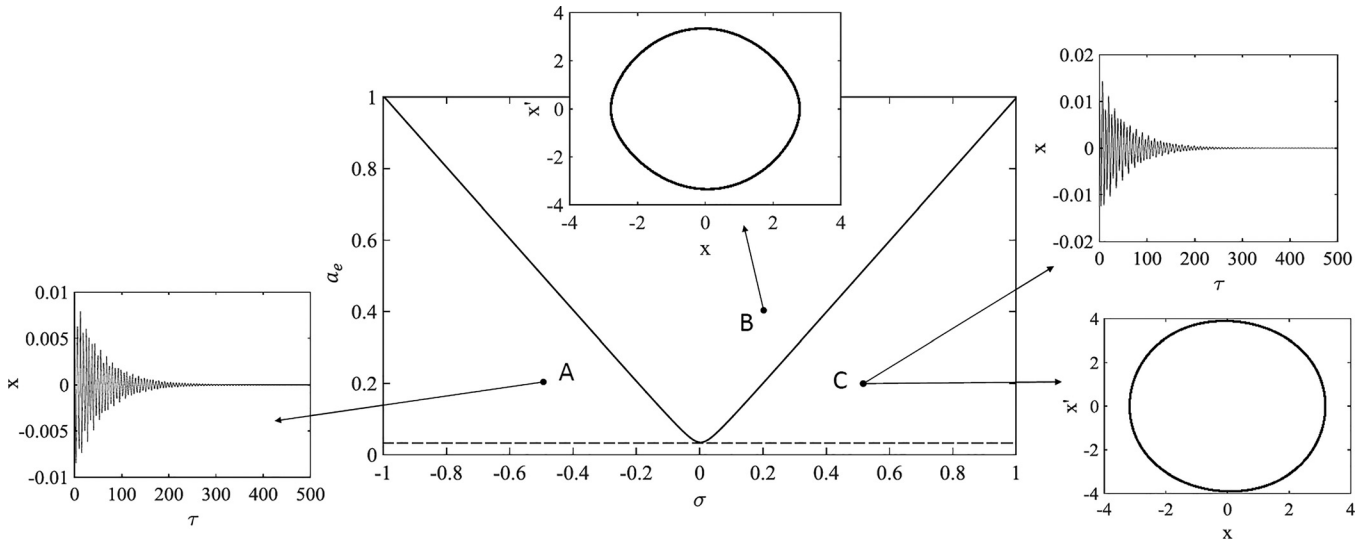

FIG. 2. Transition curves of the nonlinear parametrically excited harvester against the detuning parameter σ and excitation amplitude α_e, calculated from Eq. 10. The dashed line denotes the critical amplitude below which the response decays for any detuning. The insets show the type of numerically computed response in each region: (A) stable trivial solution; (B) stable periodic solution; (C) periodic and trivial co-existing solutions.

$$x' = -a(\omega + \sigma/2)\sin[(\omega + \sigma/2)\tau + b_0], \tag{11}$$

where b_0 is a constant phase shift. Substituting Eq. (11) into Eq. (4) allows for a solution for j to the first approximation as

$$j = \frac{a\left(\omega + \frac{\sigma}{2}\right)}{\sqrt{\rho^2 + \omega^2}} \sin\left[\left(\omega + \frac{\sigma}{2}\right)\tau + c_0\right]. \tag{12}$$

c_0 is a constant phase shift. Using Eq. (12), an expression for the average power can be extracted as

$$P_{av} = \frac{1}{T}\int_0^T j^2 \rho dt = \frac{a^2\left(\omega + \frac{\sigma}{2}\right)^2 \rho}{2(\rho^2 + \omega^2)}. \tag{13}$$

The purpose of the current work is to estimate the influence of the nonlinear coefficient β upon the frequency response of the average power output P_{av}. This is carried out in the context of a parametrically excited nonlinear system, also comparing the response curve with that of an externally excited nonlinear system, using the average power magnitude and frequency range.

The average power depends on the response amplitude a, as shown in Eq. (10). The condition for real-valued amplitudes requires the square root argument in Eq. (10) to be positive. Reviewing the form of this expression, Daqaq *et al.*[9] reported that a critical excitation amplitude exists, above which there are stable periodic solutions. This threshold can be seen in Eq. (10), which depends on the damping coefficient and the electromagnetic coupling. This expression implies that the coupling term between the mechanical and electrical systems acts as an additional dissipation mechanism. This is known to be the case in the externally excited system as well.[16] Equation (10) also provides the amplitudes for the non-trivial solution and the parameter regions in which these exist. Nayfeh and Mook discuss the extraction of the transition curves which define these regions.[8] Figure 2 reproduces these transition curves, also showing the possible response types in each region (see the insets of the figure). The frequency response of the system, shown in Fig. 3, depicts the associated average harvested power. The outcome sought is to adjust an energy harvester so that its response falls into regions B or C, where stable periodic solutions are possible.

The frequency response bears a resemblance to that of a mono-stable Duffing energy harvester. However, qualitative differences arise from the amplitude-frequency expression, Eq. (10). The nonlinear coefficient β affects the response curve *only* as a scaling factor for the amplitude. The squared response amplitude is inversely proportional to β, which implies that a parametrically excited harvester has a higher power output when β decreases (otherwise, in a weakly nonlinear regime). Furthermore, the structure of the transition curves is independent of the nonlinear coefficient. Given an excitation amplitude and damping, the bifurcation points governing the transition from regions A to B and from B to C are identical for any non-zero positive value of β. This

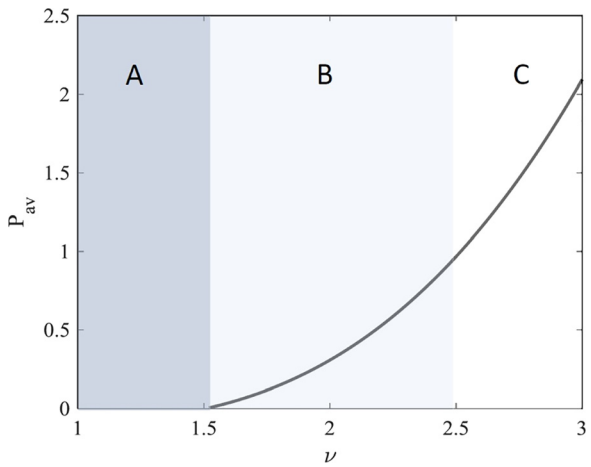

FIG. 3. Frequency response curve of the harvested power for $a_e = 0.5$. The shaded areas denote regions A, B, and C where different response types appear, as described in Fig. 2.

means that the frequency range over which a stable periodic solution would exist (regions B and C in Fig. 3) is also identical for a weakly and strongly nonlinear regime. Hence, vibration energy can be harvested over an *invariant* frequency range regardless of the value of β. This indicates that better performance would be expected with weak nonlinearity, potentially offering a solution to applications where strong nonlinearity cannot be easily realized.

It should be noted that the above discussion is based on an approximate analytical method with its accuracy restricted close to $\nu = 2$. Therefore, numerical integration of Eqs. (3) and (4) is performed for selected indicative cases, in order to impart confidence for these qualitative characteristics. Numerical integration is also performed for an externally excited Duffing-like harvester, which is identical to the parametric one of Eqs. (3) and (4), with the exception of the excitation $F_{ext} = 2\alpha_e \cos \nu\tau$ which is directly applied to the oscillator. The results are compared in order to demonstrate the advantages of parametric excitation for weakly nonlinear systems.

A relatively low excitation amplitude is chosen, $\alpha_e = 0.1$, to better represent the conditions prevalent for ambient vibrations. A reasonably low damping coefficient is used ($\gamma = 0.01$). The normalized equivalent coupling factor is $\theta = 0.35$ and $\delta = 1$ without loss of generality. Three values are considered for the nonlinear coefficients β in the ascending order (0.5, 1.0, 2.0) and the average power frequency response of the parametrically (light grey curve) and externally (dark grey curve) excited harvesters, all of which are plotted together in Fig. 4. The results show the inverse proportionality of β upon the magnitude of the average power for the parametrically excited case. Decreasing β leads to higher output power as a result of higher response amplitudes. This confirms the scaling effect of the nonlinear coefficient, observed in Eq. (10). In fact, one could expect an increase in the response amplitudes without bound as β approaches the zero limit value. This property may be extremely useful for energy harvesting purposes in the presence of weak nonlinearity, emphasising that parametrically induced vibrations can lead to higher output power. On the other hand, the externally excited system reaches a constant maximum power for all the values of β. As

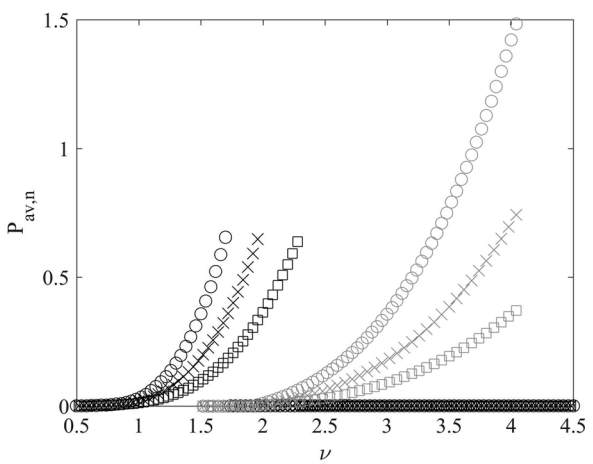

FIG. 4. Numerical integration of Eqs. (3) and (4) for $a_e = 0.1$, $\delta = 1$, $\gamma = 0.01$, $\theta = 0.35$, and $\rho = 10.6$; $\beta = 0.5$ (○), $\beta = 1$ (×), and $\beta = 2$ (□). Dark grey denotes the response of the directly excited system corresponding to $r = 0$, and light grey denotes the parametric case for $r = 1$.

long as strong nonlinearity is present, this class of harvesters is favoured since it also leads to a broad frequency range. However, this can also imply the converse argument that as β decreases, the bandwidth of the response becomes narrower, approaching essentially the response of an equivalent linear system. This is evident in Fig. 4 where it is noted that for $\beta = 2$, the bandwidth is over 1 and for $\beta = 0.5$, it is halved. In contrast, the response of the parametric system appears to extend over an invariant frequency range (extending from $\nu \approx 2$ up to $\nu \approx 4$ for all the β values considered). Most of the frequency range corresponds to region C, where the periodic solution co-exists with the trivial solution, due to the relatively low input amplitude. Overall, the transition from a strong to weak nonlinearity does not narrow the response bandwidth, and thus, broadband energy harvesting is not subject to the constraint set by it.

In this letter, parametric resonance was considered as a direct alternative to forced nonlinear energy harvesting. The response bandwidth for the latter depends, among other influential factors, on the magnitude of the nonlinear coefficient. This calls for the presence of a source of strong nonlinearity for effective broadband vibration energy harvesting. It was shown that parametrically excited systems are not subject to the same constraint since their response bandwidth is invariant of the magnitude of nonlinearity. In addition, weakly nonlinear systems lead to increased power output compared to their strongly nonlinear counterparts. These properties suggest that parametrically excited nonlinear systems are favoured alternatives for broadband vibration energy harvesting in applications where strong nonlinearity cannot be achieved.

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