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A simplified approach to the prediction of mixed and boundary friction

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ABSTRACT

Improving machine energy efficiency has led to the use of lower viscosity lubricants, and a greater chance of mixed/boundary contact occurring. A new equation, derived from experimental data, is proposed that enables simple calculation of the proportion of mixed/boundary friction in a contact, as a function of the λ ratio. It is also shown that some commonly used rough surface contact models (such as the Greenwood Tripp model) significantly underestimate mixed/boundary friction in the range $1 < \lambda < 3$. The new model has the form of an S-curve, which suggests a new approach to rough surface contact modelling is to treat it as a growth/decay process, and the consequences of this approach are discussed

1. Introduction

Improvements in energy efficiency are one of the most effective ways to reduce CO_2 emissions from machinery. In the last 20–30 years, one trend for improving energy efficiency has been to reduce the viscosity of the lubricants used in machines. Although this does result in lower viscous friction if the machine components operate in the hydrodynamic lubrication regime, lower viscosity lubricants will also result in thinner oil films, and so, particularly at higher loads and low speeds, there is a greater chance that machine components will enter the mixed and boundary lubrication regimes. Accurately predicting friction in the mixed and boundary lubrication regimes is, therefore, becoming of greater importance. Early contact models, based on pioneering insightful papers by Archard et al. [1,2], and further developed by Greenwood and Williamson [3] and later from Greenwood and Tripp [4], are still widely used today in the prediction of friction losses in the mixed and boundary lubrication regimes. The models accounted for the statistical distribution of asperity heights and predicted that critical parameters (such as friction and contact area) were proportional to applied load, even though asperities were distorted elastically. Since this early work, many other asperity models which could be used in prediction of mixed and boundary lubrication have been published [5–10]. However, these models have not always been validated against experimental data, and often contain parameters that are not straightforward to estimate. One aim of the work reported here is to propose a simple equation for predicting the amount of mixed/boundary lubrication in a contact, that is validated against experimental data, and is relatively simple to use.

For rough surface contact models, an important parameter is the lambda (λ) ratio. If the oil film thickness separating the moving rough surfaces is h (measured between the centre line averages of the rough surfaces), and the root mean square (RMS) roughnesses of the surfaces are σ_1 and σ_2 respectively, then the standard definition of λ is:

$$\lambda = \frac{h}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{1}$$

Recently, an alternative method for calculating λ has been proposed [11] which has been claimed to be an improvement on the traditional definition, particularly for “run-in” engineering surfaces. It would be a worthwhile research topic to re-examine “standard” rough surface contact models (such as those of Greenwood and Williamson [3] and Greenwood and Tripp [4]) in terms of this alternative λ value. However, in this paper, the traditional definition of λ has been used since it is the one in widespread use within the tribological community.

In general, asperity contact models do not account for the complexity of modern lubricants, in particular for lubricants that contain additives which form surface active tribo-films. Two additives are of particular importance:

(1) Anti-wear additives, of which zinc dialkyl dithiophosphate, ZDDP, is a good example. An excellent review of this additive can be found in reference [12]. ZDDP tends to form high shear surface films that can be quite thick (100–200 nm in thickness). These surface films result in high friction, and the relevant surface roughness to use (for calculating the lambda ratio, λ, which is the minimum oil film thickness

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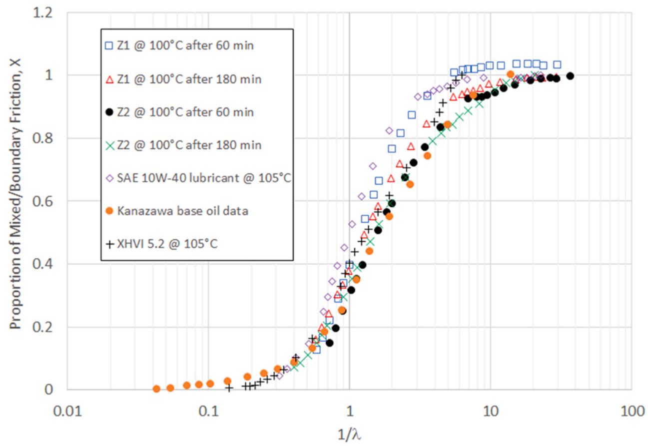

Fig. 1. The proportion of mixed/boundary lubrication, X, is plotted against 1/λ for ZDDP containing oils (oils Z1, Z2 and the SAE 10 W-40 lubricant), and base oils (Kanazawa base oil data and XHVI 5.2) to illustrate the S-curve nature of X.

divided by the combined root mean square surface roughness of the contacting surfaces) can be different from that of the metal surface that the tribo-film is deposited on.

(2) Friction modifier additives – of which glycerol mono-oleate (GMO) and molybdenum disulphide (MoS_2) are two well-known examples. A good review of friction modifiers can be found in reference [13]. These additives are thought to form molecularly thin films on the tips of asperities (which will not affect the λ ratio but can alter the relationship between stress and pressure at the asperity tip) resulting in lower friction values.

An insightful paper from the Imperial College tribology group [14] reported that if the friction coefficient in the mixed and boundary regime is normalized to its value at nominal zero film thickness and plotted against the λ ratio (as defined in Eq. (1)), both ZDDP containing lubricants, and base oils without such additives, are found to fit fairly well to a common friction curve, provided that the surface roughness of the tribo-film is included in the calculation of the λ ratio.

In this paper, additional data points are plotted to show that the concept of a common curve for different lubricants appears to be valid. The experimental data has the form of a reverse S-curve, when the proportion of mixed/boundary lubrication is plotted against λ (such curves are also known as logistic, or sigmoid curves, and naturally appear in growth/decay processes) and this insight leads to a simple equation to evaluate mixed and boundary friction coefficients at a particular λ ratio. The derived equation is convenient for tribologists and engineers to use, and it is also tested against other published experimental data on mixed/boundary friction versus λ ratio, and found to agree reasonably well with that data too.

2. Common curve for mixed/boundary friction

Recently, Dawczyk et al. [14] have shown that, based on Mini Traction Machine (MTM) data, a common normalized friction versus λ curve applies, for ZDDP containing lubricants, and for base oils.

To test this hypothesis, the authors have extended the analysis by including additional friction data from other published papers [15,16] with the data from that of Dawczyk et al. [14]. A summary plot of the various data points is shown in Fig. 1.

Note that in the graph, the vertical axis is X, defined by:

$$X = \frac{f - f_{EHD}}{f_o - f_{EHD}} \quad (2)$$

Where f is the measured friction coefficient of the lubricant, f_{EHD} is the limiting friction coefficient of the lubricant (as measured in the Mini Traction Machine) at high film thickness (i.e. at high speeds) and f_o is the maximum friction coefficient (which usually occurs at the lowest speeds). Dawczyk et al. [14] comment that X is the “fraction of load supported by boundary lubrication” and report that the total friction coefficient, f, is given by:

$$f = Xf_B + (1 - X)f_H \quad (3)$$

Where f_B is the limiting boundary friction coefficient as the oil film thickness approaches zero (which in this case is f_o) and f_H is the friction coefficient appropriate for hydrodynamic (or elastohydrodynamic) friction (which in this case is f_{EHD}). In calculating mixed/boundary friction, it is clearly of great interest to know how X, the proportion of mixed/boundary friction, varies with the λ ratio. Generally, the boundary lubrication regime would be assumed if $\lambda < 1$, whereas mixed lubrication would occur if $1 < \lambda < 3$, and the surfaces would usually be assumed to be completely separated if $\lambda > 3$.

Clearly, the contact of rough surfaces is a growth/decay process, since the proportion of mixed/boundary lubrication in a contact grows with 1/λ (from 0 % at large values of λ - when there is complete separation of the surfaces – to 100 % when $\lambda = 0$). There is a separate interdisciplinary research community that focusses on growth/decay processes, and it is well known from that research community that key parameters in such growth/decay processes will vary according to an “S-curve” (or a reverse S-curve). In particular, the proportion of mixed/boundary lubrication, X, would be expected to vary in such a fashion. To make this clear the experimental data in Fig. 1 is plotted against 1/λ and a clear S-curve variation can be seen.

Data on ZDDP containing oils Z1 and Z2 (after 60 mins and 180 min of rubbing respectively, in the Mini Traction Machine at 100 °C) were kindly supplied by the authors of reference [14], and details of these oils, and operating conditions of the Mini Traction Machine, can be found there.

Table 1 lists the materials, operating conditions, surface roughness, etc. for the various experiments discussed in this section. Three different type of experimental friction test rigs were considered. A number of researchers have published friction data using a commercially available ball on plate friction test machine known as the Mini Traction Machine [17]. Separately, friction tests were also reported by He et al. [18] that used a custom designed ball on flat friction test machine. In addition, Cui

Table 1
Summary of operating conditions, materials used, surface roughness range, and SRR, for the various experiments discussed in this section.

	Speed Range	Load	Temperature (°C)	Surface roughness range (μm)	Materials
Dawczyk et al. [14]	0.007–3.5 m/s SRR = 50 %	31 N	100	0.005–0.06	AISI 52100 steel balls & discs
Taylor [15]	0.006–2.5 m/s SRR = 100 %	19 N	105	0.005–0.06	AISI 52100 steel balls & discs
Kanazawa et al. base oils [16]	0.01–1.0 m/s SRR = 10 %	0.56 GPa	25–120	0.01–0.2	AISI 52100 steel balls & discs
He et al. [18]	10^{-4} to 35 m/s SRR = −20 %	100–700 N	40	≈0.31	Steel disc and ball
Cui et al. [19]	0–4200 rpm SRR = 200 %	0.1–1.0 Mpa	60–90	≈6.6	Friction plate: 65Mn Separator disc: 32CrMnSiA

Table 2
Summary of values of f_o and σ for the various oils plotted in Fig. 1.

	f_o	σ (nm)
SAE 10 W-40@105 °C [15]	0.127	56.6
XHVI 5.2@105 °C [15]	0.0987	5.7
Kanazawa et al. base oil data [16] (temperatures varying from 25° to 120°C)	0.199	10–200
Oil Z1@100 °C (60 mins) [14]	0.1436	56.6
Oil Z1@100 °C (180 mins) [14]	0.1394	56.6
Oil Z2@100 °C (60 mins) [14]	0.14	56.6
Oil Z2@100 °C (180 mins) [14]	0.123	56.6

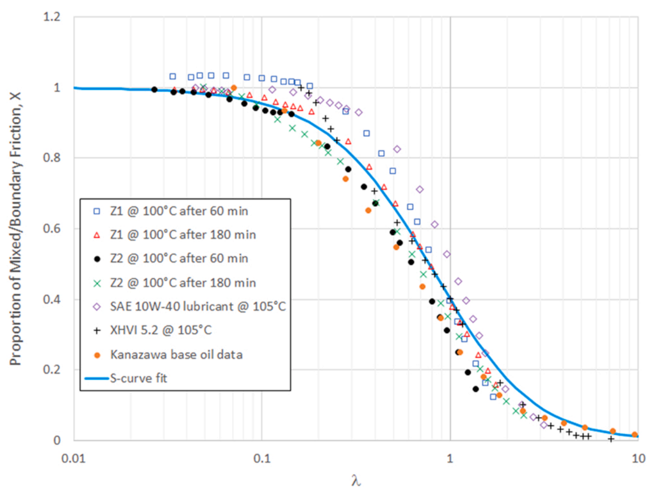

Fig. 2. Replot of Fig. 1 that includes an S-curve fit using X above, with k = 1.453 and a = 1.32. In this case the data is plotted against λ, rather than 1/λ.

et al. [19] reported asperity friction data using a custom designed rig used for a hydro-viscous drive. In Mini Traction Machine friction tests [14.–16] friction coefficients can change rapidly in the first few minutes of testing, due to the running-in of surface topography and/or the development of tribo-films, as discussed by Taylor et al. [20] and Zhang et al. [21]. For this reason, stable friction data, measured after at least 60 min of sliding on the Mini Traction Machine was used in the calculations in this paper along with surface roughness values appropriate to these "run-in" surfaces for calculation of the λ ratio. "Running-in" was not specifically discussed for the experimental tests reported by He et al. [18] and Cui et al. [19] although the surfaces were described as "well finished". It was assumed here that friction data reported in those publications [18,19] was also for run-in contacts.

It is important in friction tests to ensure there is no, or very little wear. If wear does occur, wear particles can enter the contact, as "third bodies", and cause a change in the measured friction. If the lower moving rough surface has a speed of U_1 and the upper rough surface moves at U_2, then the slide-roll-ratio (SRR) can be defined in the usual way, as SRR $= 2(U_1-U_2)/(U_1 + U_2)$. When expressed as a percentage, a slide roll ratio of 200 % corresponds to pure sliding, and an SRR ratio of 0 % corresponds to pure rolling. The SRR used in the different experiments is summarized in Table 1.

In Fig. 1, the data labelled "Kanazawa et al. base oil data" represents a high-quality recent friction dataset [16] of PAO 8 base oils (with V_k100 = 8 cSt and V_k40 = 48 cSt). These oils were tested at temperatures of 25 °C, 70 °C and 120 °C in a Mini Traction Machine using balls and disks whose composite RMS surface roughness varied between 10 and 200 nm and the average value of that dataset is plotted in Fig. 1. XHVI 5.2 is a Group III base oil (with V_k100 = 5.1 cSt and a Viscosity Index of 150). The data on the oils Z1 and Z2 is from Dawczyk et al. [14] and the SAE 10 W-40 lubricant data is available from [15]. The oils studied in the latter two papers contain ZDDP anti-wear additives. The pressure viscosity coefficients for these oils (used to calculate the elastohydrodynamic oil film thickness in the contact) were estimated using equations from Gold et al. [22].

The values of f_o and root mean square surface roughness, σ, for the various oils in Fig. 1 are summarized in Table 2 below. The larger values of σ used for the ZDDP containing oils is because, according to Dawczyk et al. [14] the tribofilms are rougher than the metal surfaces they are deposited on (at least in the case of the materials used in Mini Traction Machine tests). The surface roughness values are those reported in references [14–16] and were measured using conventional profilometers.

From Fig. 1, it is seen that some of the experimental data appears to show values for X greater than 1. This is due to the definition of X used in Eq. (2), in terms of measured friction coefficients. Using the definition of X from Eq. (3), clearly the maximum value of X is 1. However, In the experimental data, there will be small inaccuracies (errors) resulting in a spread of values (normally shown by error bars) in the friction measurement. These errors arise due to the finite accuracy of measurement of the load, speed and temperature in the friction measuring machines and can lead to limited non-physical values of X around the extremes of range. It can be seen, however, that the majority of the data in the range $0.01 < \lambda < 0.1$ (corresponding to $10 < 1/\lambda < 100$) all suggest that X is very close to 1.

It can also be seen from Fig. 1 that there is a moderately narrow span of X values. For example, at λ = 1, X is in the range 0.3–0.5 with an average value of approximately 0.4. Fig. 1 shows that the data has the classic shape of an "S-curve". As mentioned previously, S-curves, which are also known as logistic curves, or Sigmoid curves, have been widely studied [23,24] as they arise naturally in the study of growth/decay processes, of which rough surface contacts are an example. A comprehensive recent popular review of S-curves and their widespread appearance in the natural and modern world has recently been published by Smil [25]. S-curves appear in such diverse applications as: population growth, animal and organ growth, the growth (and decay) of cities, the growth of technology innovations etc. Usually, the proportion of mixed/boundary friction would be plotted against λ, rather than 1/λ, and when this is done, the graph would take the form of a "reverse S-curve". Various different mathematical functions that represent S-curves (or reverse S-curves) could be used to fit the experimental data of Fig. 1. The authors decided to use a function that has seen widespread use for fitting S-curves, and which only contains two fitting parameters. A function that contains only one fitting parameter did not provide a satisfactory fit to the experimental data, and although a better fit could be obtained using a function with three or more fitting parameters, this would make the function more complex than needed. The function used in this work is:

$$X(x) = \frac{1}{(1 + \exp(kx))^a} \tag{4}$$

Where k and a are fitting parameters. In Fig. 1 we note that the horizontal scale is logarithmic. If we set x = $\log_e \lambda$ in Eq. (4) it is found that:

$$X = \frac{1}{(1 + \lambda^k)^a} \tag{5}$$

A comparison of the measured friction data and a best fit, using the form of X above (with k = 1.453 and a = 1.32) is shown in Fig. 2. Fig. 2 has been plotted in the conventional way, in which the proportion of mixed/boundary friction, X, is plotted against λ, rather than 1/λ. The best fit values of k and a were found by minimizing the sum of the normalized squares of the differences between the fitted curve and all the data points, using Excel's Solver function. (To a good approximation, the following parameters for k and a give a good fit to the data: $k \approx \frac{3}{2}$ and $a \approx \frac{4}{3}$).

A function of similar form has previously been proposed by Olver and Spikes in 1998 [26] where, in Eq. (21) of that paper, the equation X =

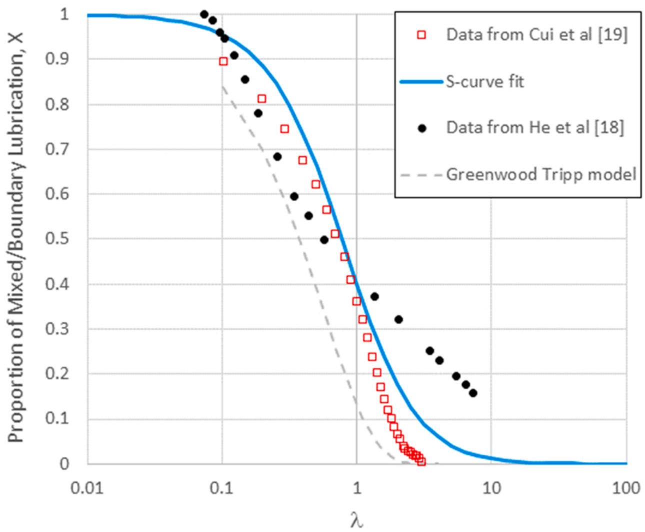

Fig. 3. Comparison of proposed new mixed/boundary friction equation (Eq. (4), with k = 1.453 and a = 1.32) with alternative experimental data from He et al. [18] and Cui et al. [19]. The value of X predicted from the Greenwood and Tripp model [4] is also shown.

$(1+\lambda)^{-m}$ was suggested for the proportion of mixed/boundary friction, with a value of m of 2, which is a special case of Eq. (5) above, when k = 1.

It should be noted that the experimental data was measured after a suitable "running-in" process had been completed, so the equation for mixed/boundary lubrication derived above (Eq. (5)) is intended to be used for fully "run-in" surfaces and is not intended to be used during the changing conditions that occur during "running-in".

The use of an S-curve to model mixed/boundary friction has the following advantages:

- There is no artificial cut-off. In many models of mixed/boundary friction, there is an assumption that the mixed/boundary friction is zero if λ is greater than a certain value (usually values of 3 or 4 are assumed).
- The gradient of X is zero at both low and high λ values.
- X is the proportion of mixed/boundary friction, and clearly increases with 1/λ (or alternatively, decreases as λ increases). In addition, it is known that the real contact area also increases with 1/λ too. S-curves often arise from well-studied generalized differential equations, and it is suggested that such an approach may be useful for the prediction of X (and/or prediction of the real contact area) and such an approach could complement previous approaches for calculating such parameters.

It could be argued that applying a mixed/boundary lubrication equation derived using data from a single tribometer, with a fairly restricted range of surface roughness, may be too ambitious, and so it is useful to compare the derived equation with other published data that reports how the proportion of mixed/boundary lubrication varies with λ. He et al. [18] have recently reported friction coefficient versus λ data from a custom designed rolling-sliding test rig of ball-on-disk design, in which the lubricant temperatures were 40 °C, the composite surface roughness of disk and ball was 0.312 μm, loads of up to 2000 N could be used, and a wide range of sliding and rolling speeds could be accommodated. In summary, this test rig is quite different to the Mini-Traction Machine tribometer from which the new mixed/boundary lubrication equation was derived. Separately, Cui et al. [19] have reported asperity friction torque data versus λ from experimental studies on a test rig that was built to analyse the torque characteristics of a hydro-viscous drive. This test rig is again quite different to the Mini-Traction Machine, and to the custom rolling-sliding test rig of He et al. [18]. Fig. 3 plots experimental data from references [18,19] compared to the proposed equation for mixed/boundary friction (Eq. (5)).

Fig. 3 shows that the proposed new mixed/boundary friction equation gives a "reasonable fit" to the experimental data from He et al. [18] and Cui et al. [19] despite the fact these experiments were carried out in experimental rigs that were quite different from the Mini-Traction Machine. For example, the S-curve fit, and the experimental data from He et al. [18] and also from Cui et al. [19] all give estimates of the proportion of mixed/boundary friction at λ = 1 of about 0.4 (i.e. around 40 %). On the other hand, the prediction from the Greenwood and Tripp model [4] is only about 0.13 (i.e. about 13 %). The S-curve fit does not fit the shape of the data from He et al. [18] that well. However, that data set is somewhat unusual, in that the friction coefficient did not show a plateau (as is normally the case) at high values of λ, so it was not possible to subtract the hydrodynamic friction from the data. The other point to make is that the error bars in mixed/boundary friction experiments are usually quite large. The spread of data in the Dawczyk et al. [14] experiments is of the order of ± 10 % (for example, at λ = 1, the proportion of mixed/boundary friction is seen, in Fig. 1, to be in the range 0.3–0.5). Unfortunately, error bars were not reported by He et al. [18] or by Cui et al. [19], but if they were of a similar size to that of the data from Fig. 1, then the S-curve fit is acceptable, and certainly a substantial improvement compared to the value predicted from the Greenwood and Tripp [4] model.

Many well-known asperity contact models predict the load carried by the asperities, and if this is divided by the total load, a value of X can be obtained which can be compared with that of the new mixed/boundary friction equation reported above, and this comparison is carried out in the next section.

3. Comparison with previous mixed/boundary friction models

Greenwood and Williamson [3] calculated the load supported by the asperities, W(λ), and the real area of contact, when a rough surface contacted a flat surface. The expression for the load supported by the asperities was of the form: W(λ) = AF(λ), where A is a pre-factor that depends on statistical details of rough surface, and also includes materials properties (such as Young's modulus) and F(λ) is a function that depends only on the λ ratio. The pre-factor A is equal to W(0)/F(0), so that W(λ)/W(0) = F(λ)/F(0). It is assumed that W(λ)/W(0) is the same as the quantity X reported in the previous section. Two expressions were reported in [3], one for the case in which the peak heights of the rough surface were distributed exponentially, and the other for the case in which the peak heights had a Gaussian distribution.

When the probability distribution function of peak heights was assumed to be an exponential distribution, Greenwood and Williamson [3] reported that:

$$X = \exp(-c\lambda) \tag{6}$$

On the other hand, if it was assumed that the probability distribution function of peak heights was a Gaussian distribution, then:

$$X = \frac{F_{3/2}(c\lambda)}{F_{3/2}(0)} \tag{7}$$

Where:

$$F_n(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty (s-u)^n . \exp\left(-\frac{s^2}{2}\right) . ds \tag{8}$$

Note that in Eqs. (6) and (7), a factor c has been introduced, to allow for the possibility that the standard deviation of the probability distribution function for peak heights could be different from the standard deviation of the probability distribution function of the overall surface. In practice, however, most engineers and tribologists, in the absence of any further information, would assume that c = 1.

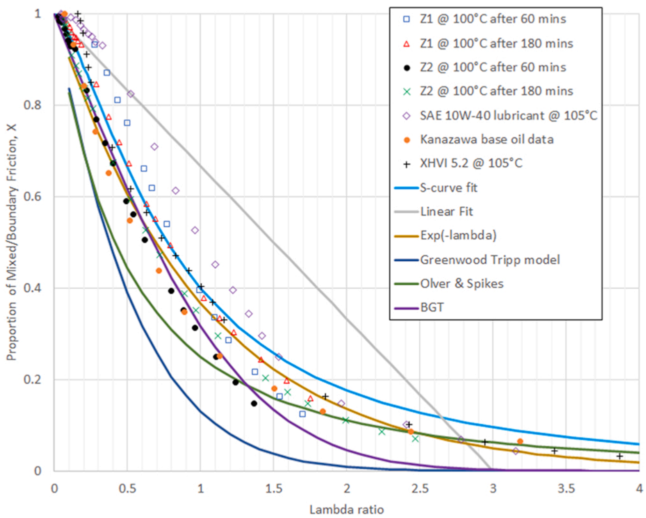

Fig. 4. Comparison of various approximations of X versus λ.

A later model, discussed by Greenwood and Tripp [4], assumed both surfaces were rough, and the following expression was derived, which is still in widespread use today:

$$X = \frac{F_{5/2}(c\lambda)}{F_{5/2}(0)} \tag{9}$$

$F_{3/2}(x)$ and $F_{5/2}(x)$ are quite complex functions, although recently an exact expression for these functions has been reported by Jedynak et al. [27].

$$F_n(\lambda) = \frac{\Gamma(n+1)}{\sqrt{2\pi}}.exp\left(-\frac{\lambda^2}{4}\right).U\left(n+\frac{1}{2}, \quad \lambda\right) \tag{10}$$

Where Γ(x) is the Gamma function, and U(m,x) is the parabolic cylinder function.

In practice, although exact equations are available for $F_n(\lambda)$, these are not that straightforward to use easily and most engineers would generally use tabulated values of these functions, at specific values of λ, and simply interpolate for intermediate λ values (tabulated values of these functions can be found in the paper by Greenwood and Tripp [4]).

As discussed in the previous section, the authors propose the use of Eq. (5) with k = 1.453 and a = 1.32.

A similar type of expression was reported by Olver and Spikes in 1998 [26]:

$$X = \frac{1}{(1+\lambda)^m} \tag{11}$$

Where the value of m was reported to be equal to 2.

Other expressions for X have also been reported, although often these apply to a restricted range of λ values. Examples include the equation below from Castro and Seabra [28] (valid for λ less than 2.1):

$$X = 1 - 0.84\lambda^{0.23} \tag{12}$$

An expression reported by Zhu et al. [29] is:

$$X = 1 - \frac{1.21\lambda^{0.64}}{1+0.37\lambda^{1.26}} \tag{13}$$

Also recently, Sander et al. [30] has approximated X, for $\lambda < 4$ by:

$$X = \left(1 - \frac{\lambda}{4}\right)^{6.804} \tag{14}$$

For $\lambda > 4$, Sander et al. [30] simply assumed that X was zero. In fact, the above expression is a good approximation for $F_{5/2}(\lambda)/F_{5/2}(0)$.

The model for asperity friction proposed by Bush, Gibson and Thomas [5] leads to:

$$X = erfc\left(\frac{\lambda}{\sqrt{2}}\right) \tag{15}$$

Where erfc(x) is the complementary error function.

A linear approximation for calculating the friction coefficient in automotive valve trains was proposed by Coy [31], and this would result in the following equation for X (for $\lambda < 3$):

$$X = \left(1 - \frac{\lambda}{3}\right) \tag{16}$$

In the above expression, if $\lambda > 3$, X = 0.

A comparison of the S-curve fit, the Greenwood and Williamson exponential fit [3], the Greenwood and Tripp fit [4], a simple linear fit [31], the Olver and Spikes fit [26] and the Bush, Gibson & Thomas fit [5], with the MTM experimental data is shown in Fig. 4. This has been plotted on a linear scale, rather than a logarithmic scale, for $0 < \lambda < 4$.

Clearly, the S-curve fit shown in Fig. 4 is a good fit to experimental friction data from the Mini Traction Machine. It can also be seen that some of the surface contact models used in mixed/boundary friction

Table 3
Predictions of X for selected λ values for the different asperity models.

λ	X					
	S-curve: $X = \frac{1}{(1+\lambda^k)^a}$ k = 1.453, a= 1.32	Exponential: $X = \exp(-\lambda)$	Linear: $X = \left(1-\frac{\lambda}{3}\right)$	Olver & Spikes [26]: $X = \frac{1}{(1+\lambda)^2}$	Greenwood-Tripp [4]: $X = \frac{F_{5/2}(\lambda)}{F_{5/2}(0)}$	BGT [5]: $X = \mathrm{erfc}\left(\frac{\lambda}{\sqrt{2}}\right)$
4	0.0594	0.0183	0	0.04	3.82E-6	6.33E-5
3.75	0.0662	0.0235	0	0.0443	1.21E-5	1.77E-4
3.5	0.0742	0.0302	0	0.0494	3.62E-5	4.65E-4
3.25	0.0838	0.0388	0	0.0554	1.03E-4	1.15E-3
3	0.0953	0.0498	0	0.0625	2.77E-4	2.70E-3
2.75	0.109	0.0639	0.0833	0.0711	7.09E-4	5.96E-3
2.5	0.127	0.0821	0.167	0.0816	1.72E-3	0.012
2.25	0.148	0.105	0.250	0.0947	3.99E-3	0.024
2	0.175	0.135	0.333	0.111	8.80E-3	0.046
1.75	0.211	0.174	0.417	0.132	0.018	0.080
1.5	0.257	0.223	0.500	0.160	0.037	0.134
1.25	0.318	0.287	0.583	0.198	0.071	0.211
1	0.401	0.368	0.667	0.250	0.131	0.317
0.75	0.513	0.472	0.75	0.327	0.230	0.453
0.5	0.663	0.607	0.833	0.444	0.390	0.617
0.25	0.848	0.779	0.917	0.64	0.636	0.803
0	1	1	1	1	1	1

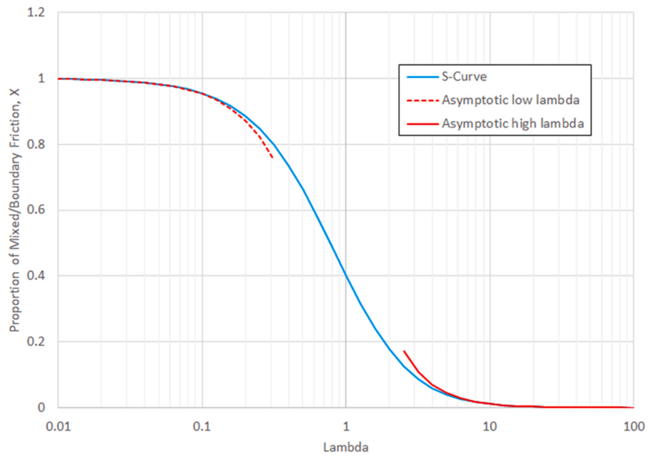

Fig. 5. Comparison of asymptotic limit expressions (for low and high λ) with full S-curve (in this case k = 1.453 and a = 1.32).

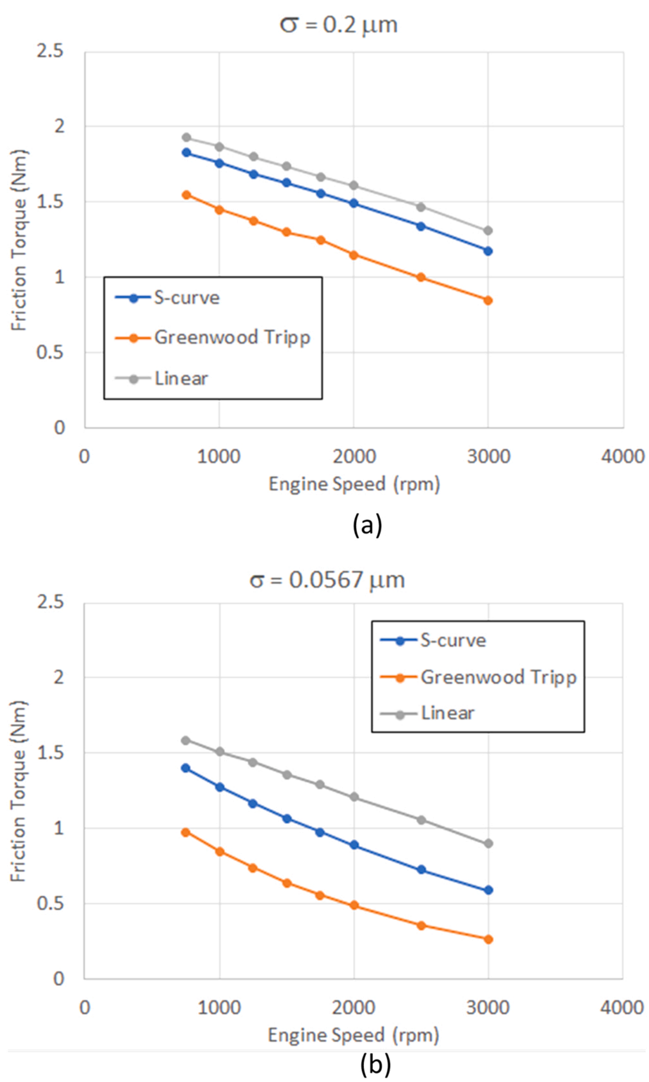

Fig. 6. (a) and (b): Predicted valve train friction torque (Nm) versus engine speed (rpm) for different mixed/boundary lubrication models. (a) shows results when the combined surface roughness is assumed to be 0.2 µm, and Friction (b) shows results for a combined surface roughness of 0.0567 µm.

prediction significantly underestimate the amount of mixed/boundary friction. For example, the widely used Greenwood & Tripp model [4] predicts a value for X of about 0.13 when λ = 1, compared to a value of 0.401 from the S-curve fit. At a value of λ = 0.1 the difference is less (with the S-curve fit giving a normalized friction of about 0.96 and the Greenwood Williamson/Greenwood Tripp models giving a value of approximately 0.84).

Of the simple models in the published literature, the one that comes closest to the S-curve fit is the exponential function, X = exp(-λ) (which predicts a value of X of 0.368 at λ = 1). The linear curve can clearly be seen to overestimate X in the mixed lubrication regime. It should also be mentioned that the Bush, Gibson & Thomas model [5] provides a good fit to the data for $\lambda \le 1$.

Table 3 lists the value of X, from the different approximations, for selected values of λ. In Table 3, the abbreviation BGT stands for Bush, Gibson and Thomas.

Table 3, and Fig. 4, suggest that both the experimental data, and many of the rough surface contact models, predict asperity friction still persists for λ > 3 (and in some cases also for λ > 4). This is due to the probabilistic nature of surface roughness. In the case of a surface with a Gaussian probability distribution, although around 99 % of asperity heights will lie between −3σ and +3σ, there is still a finite probability that there will be asperity heights larger than 3σ (or even 4σ), although of course the probability becomes smaller and smaller as asperity heights increase.

The conclusion reached here, that the Greenwood & Tripp model leads to significant underestimates of friction in the mixed/boundary lubrication regime, agrees with recent work by Leighton et al. [32]. In that work, friction was measured in a bench top reciprocating sliding test rig. It was reported that the experimentally measured mixed/boundary friction was approximately 2.75 Newtons, whereas that predicted using the Greenwood & Tripp model (assuming a surface with a Gaussian probability distribution function) was only about 2.2 Newtons. For the particular experiment in reference [32], typical λ ratios would have been expected to be less than 1.

4. Asymptotic limits of the S-curve fit

As discussed before the S-curve fit is:

$$X = \frac{1}{(1+\lambda^k)^a} \tag{17}$$

It is worth looking at the asymptotic limits of the above expression for the limits λ → 0 and λ → ∞.

For large values of λ it is clear that $X \approx \lambda^{-ak}$. On the other hand, for small values of λ

$$X \approx 1 - a\lambda^k \tag{18}$$

A comparison of these asymptotic limits with the full S-curve is shown in Fig. 5, and it is clear that the lower asymptotic limit (Eq. (18)) applies if λ < 0.3 whereas the asymptotic limit at high λ($X \approx \lambda^{-ak}$) applies if λ > 2.5.

In the high λ limit, $X \approx \lambda^{-ak}$ which for the values of k and a found for the best fit curve, would lead to $X \propto \lambda^{-1.92}$. For hydrodynamic lubrication, it is often stated [33] that $\lambda \propto (\eta U/W)^{0.5}$ (where η = dynamic viscosity (Pa.s), U = sliding speed (m/s) and W = load (N)). Therefore, with these assumptions, for high λ, X ∝ W, i.e. the proportion of mixed/boundary lubrication at high λ would be roughly proportional to applied load. A previous expression for X, due to Olver and Spikes [26] had the form of Eq. (5) with k = 1 and a = 2 and this curve would also have an asymptotic limit at high λ such that $X \propto \lambda^{-2}$. With the logistic curve found here, if it is assumed that $k \approx \frac{3}{2}$ and $a \approx \frac{4}{3}$ then ak = 2 and the relationship $X \propto \lambda^{-2}$ would hold exactly, for large λ.

Although the above discussion has focussed on X being the proportion of the load supported by the asperities, it is worth pointing out it can also be viewed as being the proportion of real contact area to the apparent contact area. Further discussion of why this is justified is outlined in Appendix A. This would suggest that at high λ values the real contact area should be roughly proportional to load, as expected.

5. A new approach to mixed/boundary friction calculation

Whenever a parameter can be described using an S-curve, it is well known that the parameter can also be related to an underlying differential equation [25]. For example, Verhulst [34], in his pioneering work in 1838 on predicting population growth, found that if N(t) is the population size at time t, and N_{max} its maximum value, then:

$$\frac{dn}{dt} \propto n(1-n) \tag{19}$$

Where $n = N(t)/N_{max}$. Since Verhulst's time, many different types of differential equations have been used for predicting S-curves, and a useful review has been published by Tsoularis [35].

It has been shown earlier in the paper that the proportion of mixed/boundary lubrication, X, varies like a reverse S-curve when plotted against λ. A suitable differential equation that can describe this behaviour is that due to Blumberg [36]:

$$\frac{dX}{d\lambda} = -\alpha X^{\beta}(1-X)^{\gamma} \tag{20}$$

Where α, β, γ are constants.

The above equation differs slightly from that reported by Blumberg [36] since a negative sign has been used on the right-hand side of the equation. This is because X decays (rather than increases) with λ.

Eq. (20) effectively states that the rate of change of X tends to zero when (1) there are very few contacts, which occurs at high values of λ and (2) when λ is small, since then the amount of mixed/boundary friction cannot change very much as it is already very close to 1.

A numerical solution of Eq. (20), using a finite difference method, which assumes that X = 0.998363 at λ = 0.01 (the value predicted by the S-curve fit) finds excellent agreement with Eq. (5) (with k = 1.453 and a = 1.32) provided that the parameters α, β, γ are: α = 1.80, β = 1.52 and γ = 0.32.

As a quick check of the above equation, when λ is large, X is small, and so in this limit Eq. (20) simplifies to become:

$$\frac{dX}{d\lambda} \approx -\alpha X^{\beta} \tag{21}$$

If it is assumed that β is approximately $\frac{3}{2}$ then the solution of the above equation is:

$$X^{-1/2} = \frac{\alpha\lambda}{2} \tag{22}$$

Therefore, X is proportional to λ^{-2} in the large λ limit, as found in the previous section.

On the other hand, when λ is small, X is close to 1, and so Eq. (20) simplifies to become:

$$\frac{dX}{d\lambda} \approx -\alpha(1-X)^{\gamma} \tag{23}$$

If it is assumed that γ is approximately equal to $\frac{1}{3}$ then the solution to the above equation is:

$$(1-X)^{2/3} = \frac{2\alpha}{3}\lambda \tag{24}$$

Which simplifies to become:

$$X = 1 - \left(\frac{2\alpha}{3}\right)^{3/2}\lambda^{3/2} \tag{25}$$

This is similar to the asymptotic limit found for small λ in the previous section.

A comparison of the asymptotic limits above with those found in Section 4 suggest that the parameters k and a (for the S-curve fit of Eq. (5)) are related to the α, β, and γ parameters in the differential equation (Eq. (20)) by:

$$\alpha = ka \tag{26}$$

$$\beta = 1 + \frac{1}{ka} \tag{27}$$

$$\gamma = 1 - \frac{1}{k} \tag{28}$$

For the previously found values of k and a (k = 1.453 and a = 1.32) this suggests that α = 1.92, β = 1.521 and γ = 0.312, which are close to the values found by numerical solution.

If it is assumed that $k \approx \frac{3}{2}$ and $a \approx \frac{4}{3}$ then $\alpha = 2$, $\beta = \frac{3}{2}$ and $\gamma = \frac{1}{3}$.

It should be emphasized that the differential equation approach to S-curves is equivalent to describing an S-curve by an analytical equation, so the friction data can either be fitted using Eq. (5), or alternatively, it can be fitted by solving the underlying differential equation (Eq. (20)).

However, since the mixed/boundary friction equation that is proposed here is based on experimental data, Eq. (20) is valid for the full range of λ values. When contact mechanics models of rough surfaces are developed, they usually assume a specific type of asperity deformation (either elastic or plastic). Such models can predict X versus λ but strictly speaking they are only valid over the λ range over which the asperity deformation assumptions hold.

In the experimental data discussed earlier, it is probable that the asperities will deform elastically at large values of λ, but it is possible that plastic deformation of asperities could occur for small values of λ. The change in friction as asperity contact conditions change is clearly accounted for in experimental data, but the authors are not aware of an analytical physics-based model that can take such asperity deformation changes into account. It is hoped that by proposing a new equation for the amount of mixed/boundary friction in a contact, based on experimental data, that it will inspire other researchers to develop a physics-based model that could predict the form of the new equation.

It should also be mentioned that the general S-curve found here (Eq. (5) and the corresponding differential Eq. (20)) are expected to be applicable to textured surfaces and rough surfaces with a non-Gaussian probability distribution function, although the values of k and a (and those of α, β and γ) are likely to be different from the values found in the paper (which are for a rough surface with a Gaussian probability distribution). Further friction experiments with surfaces that have specific surface character (i.e. height distributions) are needed to verify this assertion.

6. Application to engineering problems

The friction force due to mixed/boundary friction, F_B, for a particular lubricant, would simply be given by

$$F_B = f_o.X.W \tag{29}$$

where f_o is the friction coefficient for the lubricant (at λ = 0), X is the proportion of load carried by the asperities, and W is the total load acting on the contact. In this paper X would be calculated using the S-curve fit (for the appropriate value of λ).

The power loss due to mixed/boundary friction, P_B, would likewise be given by

$$P_B = f_o.X.W.U \tag{30}$$

where U is the relative (sliding) speed of the moving surfaces.

For low-speed contacts, it is possible to have relatively high mixed/boundary friction forces, but relatively low power losses due to mixed/

boundary lubrication. This happens, for example, in the piston assembly of an internal combustion engine, where oil films are thin close to top and bottom dead centre positions, but piston speeds are also very low (at these positions) leading to the collapse of hydrodynamic lubricating films.

As a specific example, consider the use of the work described here to predict mixed/boundary friction losses in the valve train of an internal combustion engine. The valve train is often considered, by many researchers, to be the engine component which is the major source of mixed/boundary lubrication in an engine [37–41]. Numerous researchers have measured (or predicted) valve train friction [42–47] and found it can be as high as 40 % of total engine friction at low engine speeds for a fully warmed up engine, and this would primarily all be mixed/boundary friction. The proportion of valve train friction in most modern engines is likely to be lower (due to the use of lower weight materials, and softer springs, and also because many new vehicles use stop-start systems that switch the engine off when the vehicle is stationary so that there is much less engine idling than previously), but it is still anticipated that, for passenger cars, the valve train will be the main source of mixed/boundary lubrication in an engine.

Figs. 6a and 6(b) show the predicted valve train friction torque for a typical bucket type overhead camshaft for different mixed/boundary friction models (S-curve fit, Greenwood Tripp model, and linear friction model [31]). Clearly, the predicted friction depends on many specific details of the cam profile and spring properties. For these simulations, the cam base circle was assumed to have a radius of 17.5 mm, the maximum valve lift was approximately 9 mm, the mass of the moving components in the valve train was assumed to be 0.17 kg, the preload was assumed to be 300 N, and the spring constant was assumed to be 40 N/mm. For the lubricant, the pressure viscosity coefficient was assumed to be 12 GPa^{-1}, and the lubricant viscosity was assumed to be 10 mPa.s, these values being typical of lubricants used at 100 °C. It was also assumed that there were 4 valves per cylinder, so 16 valves in total. The oil film thickness between cam and tappet was calculated using standard elastohydrodynamic line contact equations [33]. Fig. 6(a) shows that the friction torque predicted from the Greenwood Tripp model is substantially lower than that predicted using the S-curve fit proposed here. Typically, the value predicted by the Greenwood Tripp model is only about 70–80 % of the values predicted using the S-curve fit. Use of a linear fit (where $X = 1 - \lambda/3$ for $\lambda < 3$) results in slightly higher predicted valve torque. However, the linear fit does not agree well with experimental friction measurements from the MTM machine. At the lowest speeds, predicted valve train torques are close to 2 Nm (when σ, is assumed to be 0.2 μm) which is equal to a friction mean effective pressure (FMEP) of 13 kPa, and is close to measured values [37–41] (although this will vary from engine to engine, and with operating conditions, and does not include fluid film friction).

For bucket tappet valve trains, the combination of a rotating cam contacting a tappet which also rotates, generally results in quite smooth surfaces, so a typical RMS roughness would be 0.2 μm. If a value of 0.0567 μm was used (which is the RMS roughness of surfaces covered in ZDDP films, according to Dawczyk et al. [14]), lower values of friction torque would be predicted (since λ values would be higher), and the predicted valve train friction torque for this situation is shown in Fig. 6 (b). X was predicted using the S-curve fit and it was assumed that f_o was 0.12.

7. Discussion

As improvements in energy efficiency increase in importance, the move to lower viscosity lubricants is likely to continue, and this will lead to increased mixed/boundary friction in machines. Accurate prediction of the proportion of mixed/boundary friction will, therefore, become increasingly important, both for friction and wear.

The important work of Dawczyk et al. [14] has shown that both ZDDP containing oils, and base oils, can be plotted on a common curve of X versus λ (where X is the proportion of mixed/boundary lubrication) provided friction data (in this case from the Mini Traction Machine) is normalized to its value at zero film thickness, and the correct surface roughness parameters are used. In this paper, additional friction data has been analysed and it has been confirmed that this additional data also fits onto the same common curve.

A good fit to this common curve has been found with a relatively simple S-curve. A comparison of the S-curve fit with other commonly used equations for the proportion of mixed/boundary lubrication has found that some well-known (and well used) expressions (such as that due to Greenwood and Tripp [4]) lead to significant underestimates of the amount of mixed/boundary lubrication. This implies that using such models to estimate mixed/boundary friction in engineering contacts such as the valve train, may lead to significant errors and this has been demonstrated for that component using a specific case to compare the predicted mixed/boundary friction losses. Further work would be needed for surfaces that have deliberate texture, or for surfaces (such as piston liners) that contain deep honing grooves. It is possible that such surfaces may well have a common curve that is different from the one found by Dawczyk et al. [14].

Since S-curves arise from growth and decay processes, it is natural to ask whether the type of differential equations that arise in growth/decay studies can be applied to lubricated contacts. A generalized differential equation (due to Blumberg [36]) has been used to show that this approach can lead to an S-curve that is a good fit to the common curve found by the MTM measurements of Dawczyk et al. [14]. Since the differential equation is based on experimental data, it is valid for the full range of λ ratio. Other approaches to predicting the amount of mixed/boundary lubrication (or real contact area) have generally started with models that use elastic or plastic asperity deformation models, together with detailed statistical models of surfaces, to “build-up” the contact area, and/or to predict the proportion of load carried by asperities. However, strictly speaking, these models are only valid over the λ ratio where the appropriate asperity deformation mode (elastic or plastic) holds true. In this sense, a “general” differential equation based on growth/decay processes could complement these earlier “specific” models” and provide new insights.

This work has focussed on ZDDP containing lubricants and/or base oils. Further work is needed to study whether friction modified lubricants can be treated in the same fashion. This is because friction modifiers generally do not form thick tribo-film deposits. In general, it is thought friction modifier films at surfaces are only a few molecular layers thick. Therefore, friction modifier additives will not form films that are thick enough to affect the λ ratio. However, they are known to significantly reduce the friction coefficient by their influence on the shear strength at the contact. Williams [48] has suggested that such friction modifier tribo-films can alter the relationship between stress and pressure, which can directly impact how the friction coefficient would vary with λ, and inclusion of such effects may be needed for a better understanding of the friction curve when lubricants with friction modifier additives are used.

In order to validate the work reported here, it is suggested that friction experiments of the type reported in Section 2 could be repeated on specifically textured surfaces, and/or surfaces that have specific non-Gaussian surface roughness distributions. The authors would expect that the proportion of mixed/boundary friction would still vary with λ, according to Eq. (5), but that the values of k and a could well be different from those found for surfaces that have a surface roughness whose probability distribution function is Gaussian.

8. Conclusions

It has been shown that a common normalized friction curve, for both ZDDP containing oils, and base oils, originally proposed by Dawczyk et al. [14] can be fitted very well to MTM friction data from other researchers for different lubricants and can be conveniently approximated by a generalized S-curve. The curve found $X = (1 + \lambda^k)^{-a}$ (with $k \approx \frac{3}{2}$ and $a \approx \frac{4}{3}$) provides a modelling tool which is straightforward (and accurate) for engineers to use for the estimation of mixed/boundary friction compared to other equations previously reported in the literature. This curve was also found to provide a reasonable fit to recent measured asperity friction data from He et al. [18] and Cui et al. [19].

The insight that an S-curve can fit the normalized friction in the mixed/boundary lubrication regime also suggests that a "general" approach to predicting the proportion of mixed/boundary lubrication is possible, in which no assumptions are made as to whether asperity contact is plastic or elastic. A simple numerical solution of a generalized S-curve differential equation for the proportion of mixed/boundary lubrication found a good fit for the S-curve fit (which in turn was a good fit to the normalized common friction curve, which seems to fit experimental data rather well). It is also expected that such an approach would be useful for predicting the real contact area in lubricated contacts.

Application of the approach to modelling mixed/boundary friction in bucket tappet valve trains found that asperity friction models which are commonly used may significantly underestimate friction in parts of the mixed/boundary friction regimes. Consequently, a second benefit of this simplified approach may be that it also provides more accurate estimates of mixed/boundary friction than contemporary methods. The benefit of this will be to facilitate more precise prediction of power loss in internal combustion engines and other systems that operate with low viscosity lubricants.

The novelty of this work can be summarized as follows:

- A new equation that predicts the amount of mixed/boundary lubrication in a contact, as a function of λ ratio has been proposed.
- The equation was derived using an S-curve fit on measured friction data, from a Mini Traction Machine, on both unadditivated lubricants (base oils) and lubricants containing anti-wear additives (ZDDP). The common curve proposed by Dawczyk et al. [14] for ZDDP containing oils, and base oils, was found to apply to other MTM data on different oils from other researchers.
- A simple comparison of friction predictions made using the new equation with those based on other widely used asperity contact models, suggests that models, such as that due to Greenwood and Tripp [4], significantly underestimate mixed/boundary friction, particularly in the important range $1 < \lambda < 3$.
- The S-curve function that has been found to fit the common curve (found from experimental data) is a simple well-behaved function that can be used to accurately predict the proportion of mixed/boundary lubrication as λ varies.
- S-curves usually arise from growth/decay processes that are described by well-studied differential equations. The fact that an S-curve is a good fit for the common curve discussed here suggests that the type of differential equations used in growth/decay studies could be used to predict the variation of the proportion of mixed/boundary lubrication with λ, which makes no assumptions about the deformation mode of the asperities. This could complement existing "specific" approaches, where the proportion of mixed/boundary lubrication is predicted using models for given asperity deformation types and surface topography.

CRedit authorship contribution statement

All authors contributed to the study conception and design. The revised manuscript was written by Robert Ian Taylor and all authors commented on this and previous versions of the manuscript. All authors have read and approved the final manuscript.

Declaration of Competing Interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Professor Ian Sherrington reports financial support was provided by Taiho Kogyo Tribology Research Foundation.

Data availability

Data will be made available on request.

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Appendix A. : Relationship of proportion of mixed/boundary lubrication to normalized real contact area

The proportion of mixed/boundary lubrication in a contact can be defined in two different ways. Firstly, as discussed in the paper, it can be related to the load carried by the asperities. If the load carried by the asperities of the rough surfaces, when separated by a mean spacing of λ, is W(λ), then X_W can be defined as:

$$X_W = \frac{W(\lambda)}{W(0)} \tag{A1.1}$$

This has been the approach taken in much of the paper. On the other hand, the proportion of mixed/boundary lubrication can be related to the normalized real area of contact. If the real area of contact when the rough surfaces are separated by λ is $A_r(\lambda)$, then a parameter X_A may be defined as:

$$X_A = \frac{A_r(\lambda)}{A_r(0)} \tag{A1.2}$$

The question to ask is how these two different definitions are related.

If the Greenwood-Williamson model [3] with an exponential probability distribution of asperity heights is considered, then it is found that:

$$X_A = X_W = \exp(-\lambda) \tag{A1.3}$$

For the Greenwood-Williamson model [3] when a Gaussian probability distribution is assumed for the asperity heights, it is found that:

$$X_A = \frac{F_1(\lambda)}{F_1(0)}; X_W = \frac{F_{3/2}(\lambda)}{F_{3/2}(0)} \tag{A1.4}$$

For the Greenwood-Tripp model [4] when a Gaussian probability distribution is assumed for the asperity heights, it is found that:

$$X_A = \frac{F_2(\lambda)}{F_2(0)}; X_W = \frac{F_{5/2}(\lambda)}{F_{5/2}(0)} \quad \text{(A1.5)}$$

Fig. A1.1 compares the values of the functions from Eqs. (A1.4) and (A1.5). For both these models, it is found that essentially, $X_A = X_W$. Further work would be needed to check whether this relationship holds true for other rough surface contact models.

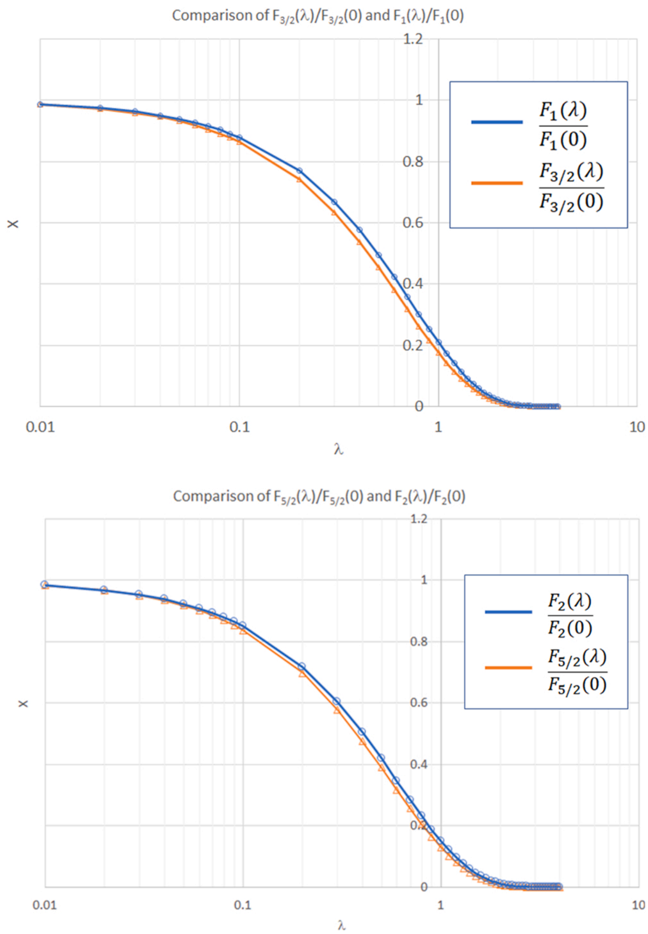

Fig. A1.1. Comparison of functions from Eqs. (A1.4) and (A1.5).

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