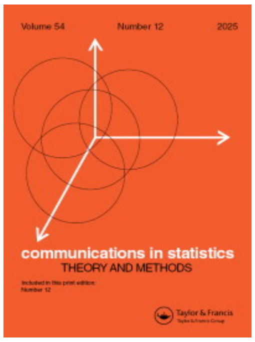

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Discrete Grönwall inequalities for demimartingales

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ABSTRACT
The aim of this work is to obtain discrete versions of stochastic Grönwall inequalities involving demimartingale sequences. The results generalize the respective theorems for martingales provided by Kruse and Scheutzow (2018) and Hendy et al. (2022).

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1. Introduction

The concept of positive association was introduced by Esary, Proschan, and Walkup (1967) and it has been studied extensively due to its applicability in many different fields such as in physics, reliability, insurance mathematics, finance, biology, etc. The definition is given below.

Definition 1.1. A finite collection of random variables $X_1, \dots, X_n$ is said to be (positively) associated if

$$\mathrm{Cov}\left(f\left(X_1, \dots, X_n\right), g\left(X_1, \dots, X_n\right)\right) \geq 0$$

for any componentwise non decreasing functions f, g on $\mathbb{R}^n$ such that the covariance is defined. An infinite collection is associated if every finite subcollection is associated.

The concept of demimartingales introduced by Newman and Wright (1982) aims, among other purposes, to study the relation between this new dependence structure with sequences of partial sums of positively associated random variables and martingales. The motivation for the definition of demimartingales was based on the following proposition which refers to mean zero positively associated random variables.

Proposition 1.2. *Suppose* $\{X_n, n \in \mathbb{N}\}$ *are* L^1*, mean zero, associated random variables, and* $S_n = \sum_{i=1}^{n} X_i$*. Then*

$$\mathbb{E}\left[\left(S_{j+1} - S_j\right) f\left(S_1, \dots, S_j\right)\right] \geq 0, \; j = 1, 2, \dots$$

for all coordinatewise non decreasing functions f.

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The definition of demimartingales presented next is a natural extension of the preceding proposition.

Definition 1.3. A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called a demimartingale if for all $j = 1, 2, \ldots$

$$\mathbb{E}\left[\left(S_{j+1} - S_j\right) f\left(S_1, \ldots, S_j\right)\right] \geq 0$$

for all componentwise non decreasing functions f whenever the expectation is defined. Moreover, if f is assumed to be non negative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called a demisubmartingale.

Based on the arguments that follow, it can be proven that a martingale with the natural choice of σ-algebras is a demimartingale.

$$\begin{aligned}\mathbb{E}\left[(S_{n+1} - S_n) f(S_1, \ldots, S_n)\right] &= \mathbb{E}\left\{E\left[(S_{n+1} - S_n) f(S_1, \ldots, S_n) \mid \mathcal{F}_n\right]\right\} \\ &= \mathbb{E}\left\{f(S_1, \ldots, S_n)\, \mathbb{E}\left[(S_{n+1} - S_n) \mid \mathcal{F}_n\right]\right\} \\ &= 0\end{aligned}$$

where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Similarly, it can be verified that a submartingale, with the natural choice of σ-algebras is also a demisubmartingale. The counterexample provided below, originally presented in Hadjikyriakou (2010), proves that the converse statement is not always true.

Example 1.4. *We define the random variables* $\{X_1, X_2\}$ *such that*

$$P(X_1 = -1, X_2 = -2) = p, P(X_1 = 1, X_2 = 2) = 1 - p$$

where $0 \leq p \leq \frac{1}{2}$. *Then* $\{X_1, X_2\}$ *is a demisubmartingale since for every non negative non decreasing function f*

$$\mathbb{E}\left[(X_2 - X_1) f(X_1)\right] = -pf(-1) + (1-p)f(1) \geq p(f(1) - f(-1)) \geq 0.$$

Observe that $\{X_1, X_2\}$ *is not a submartingale since*

$$\mathbb{E}\left[X_2 \mid X_1 = -1\right] = \sum_{x_2 = -2, 2} x_2 P(X_2 = x_2 \mid X_1 = -1) = -2 < -1.$$

It is clear by Proposition 1.2 that the partial sum of mean zero associated random variables is a demimartingale. This conclusion generalizes, in some sense, the known result that the partial sums of mean zero independent random variables form a martingale sequence. However, the counterexample implies that the class of demimartingales is a wider class of random variables compared to martingales. Moreover, counterexamples provided in Hadjikyriakou (2010) also prove that not all demimartingales have positively associated demimartingale differences, a fact that proves that demimartingales form a class of random variables wider than the class of partial sums of zero mean positively associated random variables. It worth to be mentioned that results obtained for demimartingales often generalize or even improve results available in the literature for (sub)martingales and positively associated random variables. Therefore, this new class of random objects worth to be studied independently and in depth. For more on demimartingales, we refer the interested reader to the monograph of Prakasa Rao (2012).

The Grönwall lemma appears in the literature for the first time in 1919 by T.H. Grönwall in Gronwall (1919) and since then it has been studied extensively because of its significant role in classical analysis for deriving a priori and stability estimates of solutions to differential equations (see e.g., the recent work of Hudde, Hutzenthaler, and Mazzonetto (2021) and Scheutzow (2013)). Because of its ever-increasing applications and extensions, the Grönwall inequality has been extended to include both linear and non linear cases in a general form while discrete versions of this instrumental results appear in the literature (see e.g., Alzer 1996; Yang 1988). Moreover, the study on the continuous fractional Grönwall type inequalities has been the focus for substantial research during the last decades (see e.g., Wu, Baleanu, and Zeng 2018; Ye, Gao, and Ding 2007).

Recently, Kruse and Scheutzow (2018) and Hendy, Zaky, and Suragan (2022) provided discrete versions of the stochastic Grönwall lemma involving a martingale. Considering the strong relation between martingales and demimartingales discussed above, the natural question that arises is whether similar results can be obtained for the class of demimartingales. The answer is provided in Section 2 where discrete versions of the Grönwall inequalities for demimartingales are obtained.

In what follows, the notation $x \wedge y$ represents the minimum between the real numbers x and y, while for d-dimensional vectors $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $\boldsymbol{y} = (y_1, \ldots, y_d)$ the notation $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ stands for the inner product between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. As usual, for any real number p, the p-norm of a random variable X is expressed as $\|X\|_p = \left(\mathbb{E}X^p\right)^{1/p}$. Throughout the article, we use the convention that $\sum_{j \in \emptyset} = 0$.

The next result, provided here without a proof, is crucial for obtaining the main result of this work and can be found in Christofides (2003) and Hu et al. (2010) (check also Theorem 2.1.3 in Prakasa Rao (2012)).

Lemma 1.5. *Let the sequence $\{S_n, n \geq 1\}$ be a demi(sub)martingale, $S_0 = 0$, and τ be a positive integer-valued random variable. Furthermore, suppose that the indicator function $I_{[\tau \leq j]} = h_j\left(S_1, \ldots, S_j\right)$ is a componentwise non increasing function of $S_1, \ldots, S_j$ for $j \geq 1$. Then the random sequence $\left\{S_j^* = S_{\tau \wedge j}, j \geq 1\right\}$ is a demisubmartingale.*

2. Grönwall-type inequalities for demimartingales

First, we prove a moment inequality for demimartingales. Particularly, we provide an upper bound for the p-th moment of the supremum of a demimartingale in terms of the p-th moment of its negative infimum for $0 < p < 1$. The result generalizes to the case of demimartingales the corresponding inequality for martingales proven in Kruse and Scheutzow (2018) (see Lemma 3). Athough the result is of independent interest, it will be used for the proof of Grönwall inequalities.

Throughout the section, the notation τ_n will be used to denote a stopping time for a sequence of demimartingales and a fixed x:

$$\tau_n = \max\{0 \leq k \leq n : S_k \geq x\}.$$

Proposition 2.1. *Let $S_0, S_1, S_2, \ldots$ be a demimartingale sequence such that $S_0 \equiv 0$ and $ES_{\tau_n} = 0$. Then for every $p \in (0,1)$ and every $n \in \mathbb{N}_0$ we have*

$$\mathbb{E}\left[\left(\sup_{0\le k\le n} S_k\right)^p\right] \le \frac{1}{1-p}\left(\mathbb{E}\left[-\inf_{0\le k\le n} S_k\right]\right)^p.$$

Proof. It can easily be proven that $\mathbb{E}S_j = \mathbb{E}S_{j+1}$, $\forall j$. Therefore, in the case where $S_0 \equiv 0$ we have that $\mathbb{E}S_n = 0$ for all $n \in \mathbb{N}_0$. Therefore,

$$0 = \mathbb{E}S_n = \mathbb{E}\left(S_n \vee 0\right) - \mathbb{E}\left((-S_n) \vee 0\right), \text{ for all } n \in \mathbb{N}_0$$

which leads to

$$\mathbb{E}\left(S_n \vee 0\right) = \mathbb{E}\left(-S_n \vee 0\right) \leqslant \mathbb{E}\left(\sup_{0\leqslant k\leqslant n}(-S_k)\right) = \mathbb{E}\left(-\inf_{0\leqslant k\leqslant n} S_k\right).$$

Set

$$S_j^* = S_{j\wedge\tau_n} \quad \forall j \geqslant 0.$$

Note that, for arbitrary chosen $n \in \mathbb{N}_0$, $I\{\tau_n \ge j+1\}$ is a componentwise non decreasing function of $S_1, \ldots, S_j$ and by employing Lemma 1.5, S_j^* forms a demisubmartingale sequence. Thus, for the n chosen above,

$$\left\{\sup_{0\le k\le n} S_k \ge x\right\} = \left\{\sup_{0\le k\le n} S_k^* \ge x\right\}.$$

By utilizing the Doob's type inequality for a demisubmartingale sequence, we have that for any $x > 0$,

$$xP\left(\sup_{0\le j\le n} S_j \ge x\right) = xP\left(\sup_{0\le j\le n} S_j^* \ge x\right) \le \mathbb{E}\left[S_n^* I\left\{\sup_{0\le j\le n} S_j^* \ge x\right\}\right] \le \mathbb{E}(S_n^* \vee 0).$$

Observe that

$$E(S_j^*) = E\left(S_j I\{\tau_n \ge j\}\right) + \sum_{k=0}^{j-1} E\left(S_k I\{\tau_n = k\}\right)$$

while

$$E(S_j) = E\left(S_j I\{\tau_n \ge j\}\right) + \sum_{k=0}^{j-1} E\left(S_j I\{\tau_n = k\}\right).$$

By subtracting the two latter expressions and taking into consideration that $E(S_i) = 0$, we have that

$$E(S_j^*) = \sum_{k=0}^{j-1} E[(S_k - S_j) I\{\tau_n = k\}] \ge 0$$

since $S_k \ge x$ and $S_j < x$. On the other hand $j \wedge \tau_n \le n \wedge \tau_n$ for $j = 1, 2, \ldots, n$ and by applying the demisubmartingale property for the sequence (S_j^*)

$$E(S_j^*) = E(S_{j\wedge\tau_n}) \le E(S_{n\wedge\tau_n}) = E(S_{\tau_n}) = 0$$

which leads to the conclusion that $E(S_j^*) = 0$. Thus,

$$P\left(\sup_{0\leqslant k\leqslant n} S_k \geqslant x\right) \leqslant \frac{1}{x}\mathbb{E}\left(S_n^* \vee 0\right) \leqslant \frac{1}{x}\mathbb{E}\left(-\inf_{0\le j\le n} S_j\right).$$

Then, by applying standard arguments, we have that

$$\mathbb{E}\left(\sup_{0\le j\le 2} S_j\right)^p = \int_0^\infty P\left(\sup_{0\le j\le n} S_j \geqslant x^{1/p}\right)\mathrm{d}x \le \int_0^\infty \left\{\frac{1}{x^{1/p}}\mathbb{E}\left(-\inf_{0\le j\le n} S_j\right)\right\} \wedge 1\mathrm{d}x$$

$$= \int_0^{Q^p} (Qx^{-1/p}) \wedge 1\mathrm{d}x + \int_{Q^p}^\infty (Qx^{-1/p}) \wedge 1\mathrm{d}x, \text{ where } Q = \mathbb{E}\left(-\inf_{0\le j\le n} S_j\right)$$

$$= \frac{1}{1-p}\mathbb{E}\left(-\inf_{0\le j\le n} S_j\right)^p.$$

□

As mentioned in the introduction section, the partial sums of mean zero associated random variables form a sequence of demimartingales. Thus, the corollary that follows is an immediate concequence of the proposition proven above.

Corollary 2.2. *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of mean zero associeated random variables and let $S_n = \sum_{i=1}^n X_i$. Then for every $p \in (0,1)$ and every $n \in \mathbb{N}_0$ we have*

$$\mathbb{E}\left[\left(\sup_{0\le k\le n} S_k\right)^p\right] \le \frac{1}{1-p}\left(\mathbb{E}\left[-\inf_{0\le k\le n} S_k\right]\right)^p.$$

2.1. A discrete stochastic Grönwall inequality

Next, we apply the demimartingale inequality provided in Proposition 2.1 to establish a first discrete Grönwall inequality.

Theorem 2.3. *Let $(X_n)_{n\in\mathbb{N}_0}, (F_n)_{n\in\mathbb{N}_0}$, and $(G_n)_{n\in\mathbb{N}_0}$ be sequences of non negative random variables with $\mathbb{E}[X_0] < \infty$ such that*

$$X_n \le F_n + S_n + \sum_{k=0}^{n-1} G_k X_k, \quad \text{for all } n \in \mathbb{N}_0 \tag{1}$$

where $(S_n)_{n\in\mathbb{N}_0}$ is a demimartingale such that $S_0 \equiv 0$ and $E(S_{\tau_n}) = 0$. Then, for any $p \in (0,1)$ and $\mu, \nu \in [1,\infty]$ with $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and $p\nu < 1$, it holds true that

$$\mathbb{E}\left[\sup_{0\le k\le n} X_k^p\right] \le \left(1 + \frac{1}{1-\nu p}\right)^{\frac{1}{\nu}} \left\|\prod_{k=0}^{n-1}(1+G_k)^p\right\|_\mu \left(\mathbb{E}\left[\sup_{0\le k\le n} F_k\right]\right)^p \tag{2}$$

for all $n \in \mathbb{N}_0$. If $(G_n)_{n\in\mathbb{N}_0}$ is assumed to be a deterministic sequence of non negative real numbers, then for any $p \in (0,1)$ it holds true that

$$\mathbb{E}\left[\sup_{0\le k\le n} X_k^p\right] \le \left(1 + \frac{1}{1-p}\right)\left(\prod_{k=0}^{n-1}(1+G_k)^p\right)\left(\mathbb{E}\left[\sup_{0\le k\le n} F_k\right]\right)^p \tag{3}$$

for all $n \in \mathbb{N}_0$.

Proof. The proof is motivated by the proof of Theorem 1 in Kruse and Scheutzow (2018). Let $\tilde{F}_n = \sup_{0\leq k\leq n} F_k$. Then,

$$X_n \leq (\tilde{F}_n + L_n)\prod_{i=0}^{n-1}(1+G_i),\ \forall n\in\mathbb{N}_0$$

where

$$L_n := \sum_{k=0}^{n-1}(S_{k+1}-S_k)\prod_{j=0}^{k}(1+G_i)^{-1}.$$

Set $C_k = \prod_{j=0}^{k}(1+G_i)^{-1}$. Then,

$$L_n = \sum_{k=0}^{n-1} C_k(S_{k+1}-S_k) = C_{n-1}S_n + \sum_{k=1}^{n-1}(C_{k-1}-C_k)S_k. \tag{4}$$

Observe that for any componentwise non decreasing function f we have that

$$\begin{aligned}\mathbb{E}\left[(L_{n+1}-L_n)f(L_1,\dots,L_n)\right] &= \mathbb{E}\left[C_n(S_{n+1}-S_n)f(L_1,\dots,L_n)\right]\\ &= \mathbb{E}\left[(S_{n+1}-S_n)f_1(S_1,\dots,S_n)\right]\end{aligned}$$

where $f_1(S_1,\dots,S_n) = C_n f(L_1,\dots,L_n)$. Since $G_k \geq 0$ for all integers k

$$C_{k+1} = \prod_{j=0}^{k+1}\left(1+G_j\right)^{-1} = \frac{C_k}{1+G_{k+1}} < C_k,$$

due to (4) we conclude that $f_1(S_1,\dots,S_n)$ is a componentwise non decreasing function of $S_1,\dots,S_n$. Hence, $(L_n)_{n\in\mathbb{N}_0}$ is a demimartingale. Let $\tilde{L}_n = \sup_{0\leq k\leq n} L_n$. Then,

$$\sup_{0\leqslant k\leqslant n} X_k^p \leqslant \left(\tilde{F}_n+\tilde{L}_n\right)^p\prod_{i=0}^{n-1}(1+G_i)^p.$$

By employing Holder's inequality, we have that

$$\begin{aligned}\mathbb{E}(\sup_{0\leqslant k\leqslant n} X_k^p) &\leq E\left[\left(\tilde{F}_n+\tilde{L}_n\right)^p\prod_{i=0}^{n-1}(1+G_i)^p\right]\\ &\leqslant \left[\mathbb{E}\left(\prod_{i=0}^{n-1}(1+G_i)^p\right)^{\mu}\right]^{1/\mu}\left[\mathbb{E}\left(\tilde{F}_n+\tilde{L}_n\right)^{\nu p}\right]^{1/\nu}\\ &\leq \left\|\prod_{i=0}^{n-1}(1+G_i)^p\right\|_{\mu}\left(\mathbb{E}(\tilde{F}_n^{\nu p})+\mathbb{E}(\tilde{L}_n^{\nu p})\right)^{\frac{1}{\nu}}.\end{aligned}$$

Since $(L_n)_{n\in\mathbb{N}_0}$ is a demimartingale, we can employ the result of Proposition 2.1 and get

$$\mathbb{E}(\tilde{L}_n^{\nu p}) \leq \frac{1}{1-\nu p}\mathbb{E}\left(-\inf_{0\leq j\leq n} L_j\right)^{\nu p} \leq \frac{1}{1-\nu p}\mathbb{E}(\tilde{F}_n^{\nu p}).$$

The last inequality follows since due to the non negativeness of X_n we have that $-L_n \leq \tilde{F}_n$, $\forall n \in \mathbb{N}_0$ which leads to $-\inf_{0\leq j\leq n} L_j \leq \tilde{F}_n$. Thus,

$$\mathbb{E}\left(\sup_{0\leq k\leq n} X_k\right)^p \leq \left\|\prod_{i=0}^{n-1}(1+G_i)^p\right\|_\mu \left(\mathbb{E}(\tilde{F}_n^{vp})+\mathbb{E}(\tilde{L}_n^{vp})\right)^{\frac{1}{v}}$$
$$\leq \left(1+\frac{1}{1-vp}\right)^{1/v}\left\|\prod_{i=0}^{n-1}(1+G_i)^p\right\|_\mu \left[\mathbb{E}\left(\sup_{0\leq j\leq n} F_j\right)^{vp}\right]^{1/v}$$
$$\leq \left(1+\frac{1}{1-vp}\right)^{1/v}\left\|\prod_{i=0}^{n-1}(1+G_i)^p\right\|_\mu \left[\mathbb{E}\left(\sup_{0\leq j\leq n} F_j\right)^{p}\right]$$

where the last inequality follows by applying Jensen's inequality. Inequality (3) follows by applying the same steps for $\mu = \infty$. □

Remark 2.4. It is important to highlight that, similar to the respective inequality for martingales, although the right hand side of (1) depends on a demimartingale sequence, the upper bounds provided in (2) and (3) are uniform with respect to the demimartingale sequence.

2.2. A discrete fractional stochastic Grönwall inequality

In this section, we introduce a discrete fractional Grönwall inequality involving a sequence of mean zero associated random variables. This kind of inequalities are commonly used in the numerical analysis of multi-term stochastic time-fractional diffusion equations. The physical interpretation of the fractional derivative is that it represents a degree of memory in the diffusing material.

Consider the following discretization for a time interval $[0, T]$: let τ be the temporal step-size, and let N be a positive integer such that $\tau = T/N$. Define $t_n = n\tau$, for each $n = 0, 1, \dots, N$.

The uniform $L1$-approximation for a multi-term Caputo temporal fractional derivative of orders $0 < \beta_0 < \beta_1 < \dots < \beta_m < 1$ at the time t_n is given by

$$\sum_{r=0}^{m} q_r \frac{\partial^{\beta_r} f(t)}{\partial t^{\beta_r}}\bigg|_{t=t_n} = \sum_{r=0}^{m} q_r \frac{1}{\tau^{\beta_r}\Gamma(2-\beta_r)} \sum_{i=1}^{n} a_{n-i}^{\beta_r}\left(f(t_i) - f(t_{i-1})\right) + r_\tau^n$$

where $a_j^{\beta_r} = (j+1)^{1-\beta_r} - j^{1-\beta_r}$ for each $j \geq 0$, q_r are absolutely positive parameters and r_τ^n is the truncation error (see e.g., Hendy, Zaky, and Suragan 2022 and references therein).

Definition 2.5. Let $\{f_n\}_{n=0}^N$ be a sequence of real functions. The discrete time-fractional difference operator $D_\tau^{\beta_r}$ is given by

$$D_\tau^{\beta_r} f_n = \frac{\tau^{-\beta_r}}{\Gamma(2-\beta_r)} \sum_{i=1}^{n} a_{n-i}^{\beta_r}\delta_t f_i = \frac{\tau^{-\beta_r}}{\Gamma(2-\beta_r)} \sum_{i=0}^{n} b_{n-i}^{\beta_r} f_i, \quad \forall n = 1, \dots, N$$

where $\delta_t f_i = f_i - f_{i-1}$, and the constants are defined by

$$b_0^{\beta_r} = a_0^{\beta_r}, b_n^{\beta_r} = -a_{n-1}^{\beta_r}, b_{n-i}^{\beta_r} = a_{n-i}^{\beta_r} - a_{n-i-1}^{\beta_r},$$

for each $i = 1, \dots, n-1$.

In what follows, we denote

$$\lambda = \lambda_1 + \lambda_2/(2 - 2^{1-\beta_m})$$

where λ_1 and λ_2 are positive constants and let

$$W := \sum_{r=0}^{m} q_r \frac{\tau^{1-\beta_r}}{\Gamma(2-\beta_r)} \sum_{j=1}^{k} a_{j-1}^{\beta_r} > 0.$$

Finally, recall the Mittag-Leffler function which is of the form

$$E_\alpha = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}.$$

The result that follows provides a discrete version of a fractional Grönwall inequality involving a sequence of mean zero associated random variables. The particular result is motivated by the work presented in Hendy, Zaky, and Suragan (2022) where a similar result involving a martingale sequence is presented (see Theorem 1).

Theorem 2.6. *Let $(Y_n)_{n\in\mathbb{N}}$ be mean zero associated random variables and let $(X_n)_{n\in\mathbb{N}}, (F_n)_{n\in\mathbb{N}}$ be sequences of non negative random variables with $\mathbb{E}[X_0] < \infty$ such that*

$$\sum_{r=0}^{m} q_r D_\tau^{\beta_r} X_n \le F_n + Y_n + \lambda_1 X_n + \lambda_2 X_{n-1}, \quad \forall n \ge 1 \tag{5}$$

where q_r are positive integers for $r = 0, 1, \dots, m$. Moreover, assume that $p \in (0,1)$ and $\mu, \nu \in [1, \infty]$ such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and $p\nu < 1$. Then,

$$\mathbb{E}\left[\sup_{1\le k\le n} X_k^p\right] \le \left(1 + \frac{1}{1-\nu p}\right)^{\frac{1}{\nu}} \left\| \left(2E_{\beta_m}\left(2\lambda t_n^{\beta_m}/q_m\right)\right)^p \right\|_\mu \times \left(\mathbb{E}\left[\frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)} X_0 W\right] + \mathbb{E}\left[\frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)} \sup_{1\le k\le n} F_k\right]\right)^p.$$

If $(\lambda_q)_{q\in\mathbb{N}_0}$ is a deterministic sequence of non negative real numbers, then, for any $p \in (0,1)$,

$$\mathbb{E}\left[\sup_{1\le k\le n} X_k^p\right] \le \left(1 + \frac{1}{1-p}\right) \left\| \left(2E_{\beta_m}\left(2\lambda t_n^{\beta_m}/q_m\right)\right)^p \right\|_{L^\mu(\Omega)} \times \left(\mathbb{E}\left[\frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)} X_0 W\right] + \mathbb{E}\left[\frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)} \sup_{1\le k\le n} F_k\right]\right).$$

Proof. Let $\tilde{F}_n = \sup_{1\le k\le n} F_j$. Note that since $(X_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ are sequences of non negative random variables, we employ Lemma 3 in Hendy, Zaky, and Suragan (2022) and for any integer n we write X_n as

$$X_n \le 2\left[\frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)}\left(\sum_{j=1}^{n}(F_j + Y_j) + X_0 W\right)\right] E_{\beta_m}(2\lambda t_n^{\beta_m}/q_m)$$

$$\le 2\left[\frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)}\tilde{F}_n + S_n + \frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)} X_0 W\right] E_{\beta_m}(2\lambda t_n^{\beta_m}/q_m) \tag{6}$$

where $t_n = n\tau$ and

$$S_n = \frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)}\sum_{j=1}^{n} Y_j$$

forms a demimartingale sequence due to the assumption that $(Y_n)_{n\in\mathbb{N}}$ are mean zero associated random variables. Similar to the proof of Theorem 2.3, we first apply Holder's inequality, i.e.,

$$\mathbb{E}\left(\sup_{1\le k\le n} X_k^p\right) \le \mathbb{E}\left[\left(\frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)}\tilde{F}_n + S_n + \frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)}X_0W\right)^p (2E_{\beta_m}(2\lambda t_n^{\beta_m}/q_m))^p\right]$$
$$\le \left\|(2E_{\beta_m}(2\lambda t_n^{\beta_m}/q_m))^p\right\|_\mu \left(\mathbb{E}\left(\frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)}X_0W\right)^{\nu p}\right.$$
$$\left.+\mathbb{E}\left(\frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)}\tilde{F}_n\right)^{\nu p} + \mathbb{E}(\tilde{S}_n)^{\nu p}\right)^{\frac{1}{\nu}}$$

where $\tilde{S}_n = \sup_{1\le k\le n} S_j$. Observe that, since $X_n \ge 0$ for all n, then from (6) we have that

$$-S_n \le \frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)}\tilde{F}_n + \frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)}X_0W$$

which leads to

$$-\inf_{1\le k\le n} S_k \le \frac{t_n^{\beta_m}}{q_m\Gamma(1+\beta_m)}\tilde{F}_n + \frac{\tau^{\beta_m}}{q_m\Gamma(1+\beta_m)}X_0W.$$

Proposition 2.1 is applied to the term $\mathbb{E}(\tilde{S}_n)^{\nu p}$ and the desired result follows by applying similar steps as in the proof of Theorem 2.2. □

Remark 2.7. The discrete Grönwall-type inequalities for demimartingales presented in this work can be directly applied to obtain a priori estimates for numerical schemes in stochastic analysis. For example, they may be used in the stability and convergence analysis of implicit time-stepping schemes such as the backward Euler–Maruyama method for stochastic differential equations with dependent noise structures. Moreover, the discrete fractional inequalities developed here are particularly suited for the numerical analysis of multi-term time-fractional stochastic differential equations, which arise in models with memory effects in fields such as physics, biology, and finance.

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