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REGULAR ARTICLE

Stochastic orders and shape properties for a new distorted proportional odds model

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Abstract

Building on recent developments in models focused on the shape properties of odds ratios, this paper introduces two new models that expand the class of available distributions while preserving specific shape characteristics of an underlying baseline distribution. The first model offers enhanced control over odds and log-odds functions, facilitating adjustments to skewness, tail behaviour, and hazard rates. The second model, broadening flexibility on the shape of odds functions, describes these as quantile distortions. This approach leads to an enlarged log-logistic family capable of capturing these quantile transformations and diverse hazard behaviours, including non-monotonic and bathtub-shaped rates. Central to our study are the shape relations described through stochastic orders; we establish conditions that ensure stochastic ordering both within each family and across models under various ordering concepts, such as hazard rate, likelihood ratio, and convex transform orders.

Keywords Stochastic order · Odds ratio · Log-logistic · Distortion

Mathematics Subject Classification 60E15 · 60E05 · 62E10

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1 Introduction

The development of flexible distribution models is a key pursuit in statistical research, particularly in applications that require detailed control over distributional shape properties, such as survival analysis, reliability engineering, and actuarial science. Naturally, the main goal is to introduce families of distributions allowing for fine tuning of some specific characteristic of interest. This is traditionally achieved by adding parameters to established families of distributions or by transforming characterising functionals, such as survival or hazard functions. Some recent examples of this approach are Alzaatreh et al. (2013), Kharazmi et al. (2021) or Vasconcelos et al. (2024), focusing in practical statistical properties. Recent advancements have emphasized extending these frameworks to odds functions, providing a broader and potentially more versatile approach for representing complex real-world data. A simple and flexible model preserving monotonicity and shape properties of the hazard rate is the Marshall-Olkin (MO) family of distributions, introduced in Marshall and Olkin (1997), and their extensions, whose usefulness continues to produce new applications (see Ameeq et al. 2024; Gómez-Déniz et al. 2024 or Lima et al. 2024 for some recent contributions). However, the popular proportional hazards rate (PHR) model, where the survival function is redefined by raising a baseline survival function to some power, falls beyond the MO family. On the other hand, it is well-known that increasing hazard rates lead to distributions with finite moments of every order, thus excluding the possibility of models with heavy-tails, while decreasing hazard rates seem less natural in applications. Therefore, there is convenience on broadening the approach to wider families of distributions. An alternative class of distributions, assuming increasingness of the odds function, has been recently studied by Lando et al. (2022) and Zardasht (2024). This class includes every distribution with increasing hazard rate, and allows for heavy-tailed distributions and some bathtub-shaped hazard rates. Therefore, it is quite natural to explore models where we are interested in the proportionality of the odds function. Examples of model construction based on this idea have been followed by Bennett (1983); Collett (2023); Dinse and Lagakos (1983, 1984); Rossini and Tsiatis (1996) or Panja et al. (2024), while structural properties of such models have been studied in Kirmani and Gupta (2001) or Sankaran and Jayakumar (2008)

Bearing in mind the need to add flexibility to shape properties of the odds function, namely, monotonicity, increasing rate or degree of convexity, we propose two different models that address the preservation of these characteristics. Being interested in shape properties expressed through the odds, we are naturally driven into studying structural relations expressed by stochastic ordering properties, instead of a more statistical and computational approach. We note that a similar approach has been studied in Cordeiro et al. (2014), concentrated on the control of the tail behaviour of the distributions, and in the statistical properties and estimation aspects, instead of the ordering relationships. The first model we propose, that we name the *odds-Marshal-Olkin* (oMO), adapts the proportionality idea to the odds function. In fact, the oMO model encompasses the proportionality of the odds and also of the log-odds, providing more flexibility, as applying a logarithmic transformation often leads to an affine relation. The usefulness of this adaptability is illustrated in Example 1 below. However, the

oMO model excludes the already mentioned PHR model, that, in general, produces a more complex odds function. Moreover, the oMO model, although attractive due to its simplicity, allows for limited preservation of interesting stochastic ordering relations or of shape properties. A second and more general construction, that we name *distorted odds-Marshal-Olkin model* (d-oMO), addresses these difficulties and shows a richer stochastic ordering structure. This broader model defines the odds function by transforming a baseline odds function through the quantile function of an enlarged log-logistic family of distributions (see Definition 5.1 below for details). This relation means that ordering relations within this new family of distributions translate into the d-oMO distributions, raising the interest in exploring also the properties of this enlarged log-logistic family.

The paper is structured as follows. Section 2 provides preliminary definitions and background essential for understanding the proposed models, including a review of stochastic orders and shape properties in distribution families. In Sect. 3, we introduce the oMO model, with a focus on its properties and stochastic comparisons with the baseline distribution. Section 4 extends the model by defining a distorted odds ratio model, the d-oMO model, leveraging additional parameters to further control distributional shape. Finally, in Sect. 5, we explore the enlarged log-logistic distribution, detailing its implications for odds and hazard rate behaviours.

2 Preliminaries and basic definitions

We shall represent by X, F and f the baseline random variable, its cumulative and density functions (that we will be assuming to exist), respectively. Analogously, Y, G and g, possibly with some subscripts to denote parameters, will represent the new models to be studied. Moreover, survival functions are represented as $\overline{F}(x) = 1 - F(x)$ or $\overline{G}(x) = 1 - G(x)$. We shall refer to the random variables or to their distribution functions as is more convenient. In fact, the characterisations we will be discussing depend only on the distribution, so the random variables will appear only as a convenience. We recall the usual notions which were briefly mentioned in the Introduction. Given a distribution function F, its hazard rate and reversed hazard rate are denoted with $h_F(x) = \frac{f(x)}{\overline{F}(x)}$ and $\widetilde{h}_F(x) = \frac{f(x)}{F(x)}$ respectively, its odds function with $\Lambda_F(x) = \frac{F(x)}{\overline{F}(x)} = \frac{1}{\overline{F}(x)} - 1$, and its odds rate with $\lambda_F(x) = \Lambda'_F(x) = \frac{f(x)}{\overline{F}^2(x)}$. While the monotonicity of the hazard rate function has been extensively studied in the literature, for the odds function, which is always increasing, the interest relies on its growth rate, characterised by monotonicity of λ_F. These functions may be used to define some classes of distributions.

Definition 2.1 We say that X or F have

1. Increasing (decreasing) hazard rate, represented by $F \in \mathrm{IHR}(\ F \in \mathrm{DHR})$, if h_F is increasing (decreasing);

2. Increasing (decreasing) odds rate, represented by $F \in \text{IOR}$ ($F \in \text{DOR}$), if λ_F is increasing (decreasing);
3. Convex (concave) log-odds if $\log \Lambda_F(x)$ is convex (concave).

The IHR and DHR families are well-known in the literature, while the IOR family has been receiving less attention. Some properties of the IOR class are studied in a systematic way in Lando et al. (2022). The DOR family is only briefly mentioned in Arab et al. (2024), and, also recently discussed in Chen et al. (2024), although with a different terminology. Note that $F \in \text{IOR}$ ($F \in \text{DOR}$) is equivalent to the odds function Λ_F being convex (concave), so these odds rate classes describe a shape property of the corresponding distributions.

We now recall some common stochastic order notions that will be considered later.

Definition 2.2 Consider two distribution functions F_1 and F_2, with densities f_1 and f_2, respectively. We say that F_1 is smaller than $F_2$1. In the usual stochastic order, denoted as $F_1 \leq_{st} F_2$, if $\overline{F}_1(x) \leq \overline{F}_2(x)$, for every $x \in \mathbb{R}$;
2. In the hazard rate order, denoted as $F_1 \leq_{hr} F_2$, if $h_{F_1}(x) \geq h_{F_2}(x)$, for every $x \in \mathbb{R}$;
3. In the reversed hazard rate order, denoted as $F_1 \leq_{rh} F_2$, if $\widetilde{h}_{F_1}(x) \geq \widetilde{h}_{F_2}(x)$, for every $x \in \mathbb{R}$;
4. In the likelihood rate order, denoted as $F_1 \leq_{lr} F_2$, if $\frac{f_2(x)}{f_1(x)}$, is increasing.
5. In the dispersive order, denoted as $F_1 \leq_{disp} F_2$, if $F_2^{-1} \circ F_1(x) - x$ increases in x.

Given the alternative expression for the odds function, the following statement is straightforward.

Proposition 2.3 *Given two distribution functions F_1 and F_2, $F_1 \leq_{st} F_2$ if and only if $\Lambda_{F_1}(x) \geq \Lambda_{F_2}(x)$.*

The classes mentioned in Definition 2.1 defined by the monotonicity of the hazard or the odds rate may be characterised via a different type of stochastic order, namely the convex transform order which involves a shape restriction on the transformation that maps one distribution to the one being compared.

Definition 2.4 [van Zwet (1964)] Given two distribution functions F_1 and F_2, we say that F_1 is smaller than F_2 in the convex transform order, represented by $F_1 \leq_c F_2$, if $F_2^{-1} \circ F_1$ is convex.

Let us now fix, for the sequel, two reference distributions: the standard exponential, with distribution function $\mathcal{E}(x) = 1 - e^{-x}$, and the standard log-logistic, with distribution function $\mathcal{L}(x) = 1 - \frac{1}{x+1} = \frac{x}{x+1}$. It is well-known that $F \in \text{IHR}$ ($F \in \text{DHR}$) if and only if $F \leq_c \mathcal{E}$ ($F \geq_c \mathcal{E}$). Analogously, as referred in Lando et al. (2022), it is easily seen that $F \in \text{IOR}$ ($F \in \text{DOR}$) if and only if $F \leq_c \mathcal{L}$ ($F \geq_c \mathcal{L}$).

For a systematic study of properties of the stochastic orders defined above, and a number of other interesting stochastic order relations, and relations among them, we refer the interested reader to the monographs (Shaked and Shanthikumar 2007) or (Marshall and Olkin 2007).

3 The odds-Marshall-Olkin model

The study of the growth rate of the odds function is fundamental in the characterisation of distribution families that maintain specific shape properties such as the IOR, particularly in reliability and survival analysis. In this context, we introduce a modified proportional odds model that leverages the properties of the IOR and log-odds convexity to create new distribution families. This approach extends the Marshall-Olkin method to the construction of families of distributions to the broader proportional odds framework.

Definition 3.1 Let $\beta, \theta > 0$. Given a baseline distribution function F, we define the *odds-Marshall-Olkin* (oMO) distribution function $G_{\beta,\theta}$ by

$$\Lambda_{G_{\beta,\theta}}(x) = \beta \Lambda_F^{\theta}(x) = \beta \left(\frac{F(x)}{\overline{F}(x)} \right)^{\theta}. \tag{1}$$

It is obvious that $G_{\beta,1}$ has odds function $\Lambda_{G_{\beta,1}}$ proportional to Λ_F, while for $G_{1,\theta}$ we have that $\log \Lambda_{G_{1,\theta}}(x) = \theta \log \Lambda_F(x)$, that is, (1) covers the case of a model with proportional log-odds. The classical Marshal-Olkin model, for which there exists a huge literature, is obtained by taking $\theta = 1$. It is also clear from (1) that the oMO model encompasses affine relations of the odds function with respect to the baseline distribution, thus going beyond the strict proportionality.

Taking into account that $\Lambda_{G_{\beta,\theta}}(x) = \frac{G_{\beta,\theta}(x)}{\overline{G}_{\beta,\theta}(x)} = \frac{1}{\overline{G}_{\beta,\theta}(x)} - 1$, it follows easily that, for each $x \in \mathbb{R}$,

$$G_{\beta,\theta}(x) = \frac{\beta F^{\theta}(x)}{\beta F^{\theta}(x) + \overline{F}^{\theta}(x)}, \qquad \text{and} \qquad \overline{G}_{\beta,\theta}(x) = \frac{\overline{F}^{\theta}(x)}{\beta F^{\theta}(x) + \overline{F}^{\theta}(x)}. \tag{2}$$

The following shape characterisation, for $\theta = 1$, is a straightforward consequence of the convexity properties of the function $\frac{\beta x}{1-(1-\beta)x}$ when $x \in [0,1]$.

Proposition 3.2 *If F is concave, then $G_{\beta,1}$ for $\beta \geq 1$ is also concave. If F is convex, then $G_{\beta,1}$ for $\beta \leq 1$ is also convex.*

It is easily seen that the corresponding transformation for $\theta \neq 1$ and general $\beta > 0$ is neither convex nor concave, so no conclusion about the convexity of $G_{\beta,\theta}$ can be drawn.

From (2), the density and hazard rate functions for $G_{\beta,\theta}$ are easily obtained:

$$g_{\beta,\theta}(x) = \beta\theta f(x)\frac{F^{\theta-1}(x)\overline{F}^{\theta-1}(x)}{(\beta F^{\theta}(x) + \overline{F}^{\theta}(x))^2}, \tag{3}$$

and

$$h_{G_{\beta,\theta}}(x) = \beta\theta h_F(x)\frac{F^{\theta-1}(x)}{\beta F^{\theta}(x) + \overline{F}^{\theta}(x)} = \beta\theta h_F(x)\left(\frac{F(x)}{\overline{F}(x)}\right)^{\theta-1}\frac{\overline{G}_{\beta,\theta}(x)}{\overline{F}(x)}. \tag{4}$$

The effect of the parameters on the density of $G_{\beta,\theta}$ is illustrated in Fig. 1. The parameter θ concentrates the distribution while creating lighter tails, whereas β shifts the concentration region to the left, skewing the distribution to the right and making the tail slightly lighter. Obviously, the interplay between the parameters provides control over spread and tail behaviour, offering an increased adaptability in applications.

Defining $T_{\beta,\theta}(x) = \beta\theta\frac{x^{\theta-1}}{\beta x^{\theta}+(1-x)^{\theta}}$, the first equality in (4) may be rewritten as

$h_{G_{\beta,\theta}}(x) = h_F(x)T_{\beta,\theta}(F(x))$, implying immediate shape characterisations for some distributions included in the oMO model.

Theorem 3.3 1. If $F \in$ IHR and $\beta \leq 1$, then $G_{\beta,1} \in$ IHR.
2. If $F \in$ DHR and $\beta \geq 1$, then $G_{\beta,1} \in$ DHR.
3. If $F \in$ IOR, then, for $\theta \geq 1$, $G_{\beta,\theta} \in$ IOR.

Proof The result is immediate once we verify the monotonicity properties of $T_{\beta,\theta}$. It is easily verified that $T_{\beta,\theta}$ is monotone only for $\theta = 1$, that $T_{\beta,1}$ is increasing for $\beta \leq 1$, and that $T_{\beta,1}$ is decreasing for $\beta \geq 1$. With regard to the preservation of the IOR property, it follows directly from (1) taking into account that $\theta \geq 1$. □

Note that, as for Proposition 3.2, when $\theta \neq 1$, no conclusion can be drawn about the monotonicity of the hazard of $G_{\beta,\theta}$, as the corresponding $T_{\beta,\theta}$ is not monotonous.

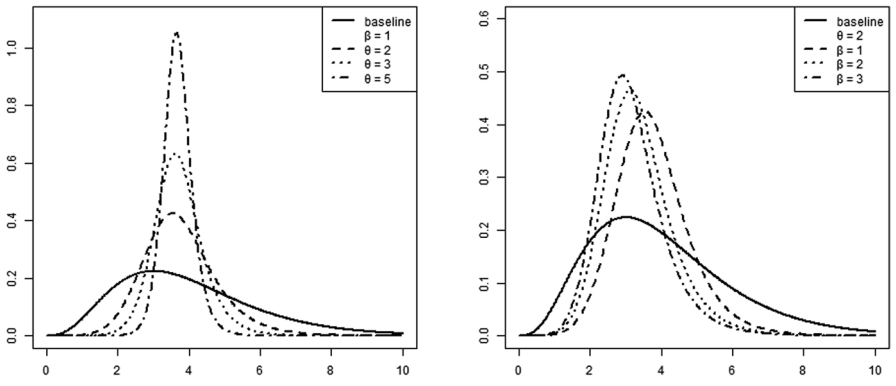

Fig. 1 Densities of the baseline distribution $\Gamma(4, 1)$ and $G_{\beta,\theta}$

However, shifting our interest for the odds rate, part 3. in Theorem 3.3 provides results for $\theta \geq 1$. Therefore, the odds rate shows greater flexibility in capturing the varying behaviours of model (1).

Given that $G_{\beta,\theta}$ is a transformation of F, it is natural to compare the baseline with the transformed distribution. Specifically, we are interested in understanding how the transformation applied to F affects key reliability properties and relationships.

Theorem 3.4 1. If $\beta \leq 1$, then $F \leq_{lr} G_{\beta,1}$.
2. If $\beta \geq 1$, then $F \geq_{lr} G_{\beta,1}$.

Proof It is easily verified that $\frac{g_{\beta,1}(x)}{f(x)} = \frac{\beta}{(1+(\beta-1)F(x))^2}$, so the result follows immediately. □

Corollary 3.5 1. If $\beta \leq 1$, then $F \leq_{rh} G_{\beta,1}$, $F \leq_{hr} G_{\beta,1}$ and $F \leq_{st} G_{\beta,1}$.
2. If $\beta \geq 1$, then $F \geq_{rh} G_{\beta,1}$, $F \geq_{hr} G_{\beta,1}$ and $F \geq_{st} G_{\beta,1}$.

Proof This is an immediate consequence of Theorem 3.4 and Theorem 1.C.1 in Shaked and Shanthikumar (2007). □

Note that when $\theta = 1$, the following explicit bounds for $h_{G_{\beta,1}}$ are immediate:

1. For $\beta \leq 1$, $\beta h_F(x) \leq h_{G_{\beta,1}}(x) \leq h_F(x)$,
2. For $\beta \geq 1$, $h_F(x) \leq h_{G_{\beta,1}}(x) \leq \beta h_F(x)$.

The previous results mention comparability for $G_{\beta,1}$. This choice for θ is the only one allowing for comparability results, as stated next.

Corollary 3.6 *For $\theta \neq 1$, F and $G_{\beta,\theta}$ are not comparable with respect to the usual stochastic order. Therefore, they are also not comparable with respect to $\leq_{hr}$, $\leq_{rh}$ or $\leq_{lr}$ stochastic orders.*

Proof We need to look at

$$\overline{G}_{\beta,\theta}(x) - \overline{F}(x) \stackrel{\text{sgn}}{=} \frac{\overline{F}(x)^{\theta-1}}{\beta F(x)^{\theta} + \overline{F}(x)^{\theta}} - 1,$$

so, the conclusion follows by analysing the sign of $S_{\beta,\theta}(x) = \frac{(1-x)^{\theta-1}}{\beta x^{\theta}+(1-x)^{\theta}} - 1$, for $x \in [0,1]$. Differentiating, one finds $S'_{\beta,\theta}(x) \stackrel{\text{sgn}}{=} (1-x)^{\theta} - \beta x^{\theta-1}(\theta - x)$. For $\theta \neq 1$, one has $S_{\beta,\theta}(0) = 0$, $S'_{\beta,\theta}$ is positive for x near 0 if $\theta > 1$, and is negative if $\theta < 1$. Finally, noting that $S_{\beta,\theta}(1) = -1$ if $\theta > 1$, and $S_{\beta,\theta}(1) = +\infty$ if $\theta < 1$, the first part of the result is proved. The second part is a consequence of Theorem 1.C.1 in Shaked and Shanthikumar (2007) □

Example 1 As an example of the usefulness of model (1), we consider the data of Veteran's Administration lung cancer trial reported by Prentice (1973), that was analysed using a proportional odds model by Bennett (1983). The data describes the survival days of the 97 patients that had no prior treatment and two covariates, a performance status and tumor type. The analysis separated patients in two groups: low performance status (less or equal to 50) and high performance status. As noted in Bennett (1983), the log-odds of the two groups have, approximately, a constant difference, suggesting an affine relation, which served as a justification to apply a methodology based on a proportional odds model. However, running a linearity approximation test between the two sets of odds or log-odds clearly indicates that a linear relation between the odds can only explain about 75% of the variability observed in the data, while assuming the linearity of the log-odds, one can explain about 90% of the variability. Moreover, the estimates of the linear coefficients clearly suggests using model (1) with $\beta = 4.4324$ and $\theta = 0.6822$. Besides, the group with high performance status has 72 observations, while the low performance status only contains 25 points. Therefore, to estimate the distribution, it is convenient to take as baseline distribution the one describing the high performance status group and then use the parameters mentioned above to get an estimate for the distribution of the survival days for the low performance status group. For the high performance status an estimate suggests a $\Gamma(1.13, 116.6)$. Hence, the density function for the survival of the low performance status group is approximated by

$$g_{4.4324,0.6822}(x) = 4.4324 \times 0.6822 \frac{f(x)F^{-0.3178}(x)\overline{F}^{-0.3178}(x)}{\left(4.4324F^{0.6822}(x) + \overline{F}^{0.6822}(x)\right)^2}$$

where $f(x)$ and $F(x)$ are, respectively, the density and distribution functions of the $\Gamma(1.13, 116.6)$ distribution. Taking into account Theorem 3.3, although the particular F is IOR, we cannot derive the same property about $G_{4.4324,0.6822}$.

Note that, by construction, the oMO model leads to ordering or shape properties only for $\theta = 1$. Although the underlying motivations were of a different nature, the model discussed in the next section allows for results with $\theta \neq 1$.

4 The distorted odds-Marshall-Olkin model

The previous section studied stochastic ordering relations between distributions defined by a specific transformation of the odds function, having in mind the possibility of mixing the proportionality of the odds ratio and of the log-odds ratio, each controlled by an appropriate parameter. Observe that the odds function of the $G_{\beta,\theta}$ distribution appears as a distortion of the odds ratio Λ_F of the baseline distribution function F, adding a layer of flexibility in modelling the shape of the odds function. The model introduced in Definition 3.1 mimics the construction of the PHR model, where a survival function is transformed into $\overline{F}^{\theta}(x)$ for some $\theta > 0$. However, the PHR model is not covered by the family of distributions introduced in Definition 3.1.

In fact, the odds function for the PHR model is of the form $(1+\Lambda_F(x))^\theta - 1$. It is worth noting that this latter form corresponds to transforming the underlying distribution F by the quantile function of a Pareto distribution with survival function $(x+1)^{-\frac{1}{\theta}}$. This observation will be explored later in Sect. 5 in more generality. Nevertheless, the general form of the odds function for the PHR model suggests an extension of the transformation used in Definition 3.1, targeted at fine tuning the tail behaviour of the odds rate.

Definition 4.1 Let $\alpha \geq 0$, β, $\theta > 0$. Given a baseline distribution function F, we define the distorted odds-Marshall-Olkin (d-oMO) distribution functions $G_{\alpha,\beta,\theta}$ by

$$\Lambda_{G_{\alpha,\beta,\theta}}(x) = \beta\left((\alpha+\Lambda_F(x))^\theta - \alpha^\theta\right). \tag{5}$$

It is obvious that the model introduced in Definition 3.1 is a particular case of (5), taking $\alpha = 0$, while the PHR model is obtained by choosing $(\alpha,\beta,\theta) = (1,1,\theta)$. Moreover, note that this extended model unifies the proportionality models we have discussed (odds, log-odds and hazard rate), offering seamless transition between them.

Taking into account that $\Lambda_{G_{\alpha,\beta,\theta}}(x) = \frac{1}{\overline{G}_{\alpha,\beta,\theta}(x)} - 1$, the following explicit representation for the distributions $G_{\alpha,\beta,\theta}$ introduced in Definition 4.1 is immediate:

$$\overline{G}_{\alpha,\beta,\theta}(x) = \frac{1}{1+\beta((\alpha+\Lambda_F(x))^\theta - \alpha^\theta)}, \tag{6}$$

while the density function is represented as

$$g_{\alpha,\beta,\theta}(x) = \frac{\beta\theta(\alpha+\Lambda_F(x))^\theta}{\overline{F}(x)(\alpha\overline{F}(x)+F(x))}\overline{G}^2_{\alpha,\beta,\theta}(x)f(x) = \beta\theta(\alpha+\Lambda_F(x))^{\theta-1}\frac{\overline{G}^2_{\alpha,\beta,\theta}(x)}{\overline{F}(x)}h_F(x).$$

Remark 4.2 The distribution function $G_{\alpha,\beta,\theta}$ can be written as

$$G_{\alpha,\beta,\theta}(x) = \frac{\beta\left((\alpha+(1-\alpha)F(x))^\theta - (\alpha\overline{F}(x))^\theta\right)}{\overline{F}^\theta(x) + \beta\left((\alpha+(1-\alpha)F(x))^\theta - (\alpha\overline{F}(x))^\theta\right)}. \tag{7}$$

In the special case where $\alpha = 1$, $G_{1,\beta,\theta}$ is the recently defined MPHR model introduced in Balakrishnan et al. (2018). Das and Kayal (2021) later extended this model by incorporating a scale parameter, calling it MPHRS. Similarly, our models can be generalised by introducing a scale parameter in the same way.

Remark 4.3 The family of distributions $G_{\alpha,\beta,\theta}$ depends on three parameters. Here is a brief description of the effect of each one of them. All the parameters, as they increase, shift the mass towards the origin, introducing skewness and lighter tails. The parameter β has a small effect on the mass shifting, affecting mainly mass con-

centration. The parameter θ has a dramatic effect in both shifting the mass closer to the origin and the concentration (take into consideration the vertical scale of the plots), hence contributing very significantly to skewness and lighter tails. An illustration of these effects can be found in Fig. 2.

Although it seems that the $G_{\alpha,\beta,\theta}$ family is not closed under formation of maximums, that is, in general the distribution of the form $G^n_{\alpha,\beta,\theta}(x)$, $n \geq 2$, is not included in the d-oMO model, we may still find an *extreme geometrical stability property* (see Marshall and Olkin 1997).

Theorem 4.4 *Let $X_1, X_2, \ldots$ be independent and with distribution function $G_{\alpha,\beta,\theta}$, for some fixed values of $\alpha \geq 0$, $\beta, \theta > 0$, and consider N, independent from the X_n, with geometric distribution, $P(N = n) = p(1-p)^{n-1}$, $n \geq 1$, for some $p \in [0, 1]$. Define $U = \min\{X_1, \ldots, X_N\}$ and $V = \max\{X_1, \ldots, X_N\}$. Then, the distribution function of U and V are $G_{\alpha,\frac{\beta}{p},\theta}$ and $G_{\alpha,\beta p,\theta}$, respectively. Or, equivalently, the family of distributions $G_{\alpha,\beta,\theta}$ has geometric extreme stability.*

Proof Proceeding by conditioning, the distribution function of U is

$$\overline{F}_U(x) = \sum_{n=1}^{\infty} \overline{G}^n_{\alpha,\beta,\theta}(x) p(1-p)^{n-1} = \frac{p\overline{G}_{\alpha,\beta,\theta}(x)}{1-(1-p)\overline{G}_{\alpha,\beta,\theta}(x)}.$$

Using now the representation for $\overline{G}_{\alpha,\beta,\theta}$ that follows from (7), the result is immediate. The case of V is treated analogously. □

4.1 Preservation of monotonicity properties

Given the expressions above, we have the following representation for the hazard rate function for the distributions in the d-oMO model:

$$h_{G_{\alpha,\beta,\theta}}(x) = \frac{g_{\alpha,\beta,\theta}(x)}{\overline{G}_{\alpha,\beta,\theta}(x)} = \frac{\beta\theta(\alpha+\Lambda_F(x))^{\theta-1}}{1+\beta\left((\alpha+\Lambda_F(x))^{\theta}-\alpha^{\theta}\right)} \cdot \frac{h_F(x)}{\overline{F}(x)} = \beta\theta h_F(x) T_{\alpha,\beta,\theta}(\Lambda_F(x)),$$

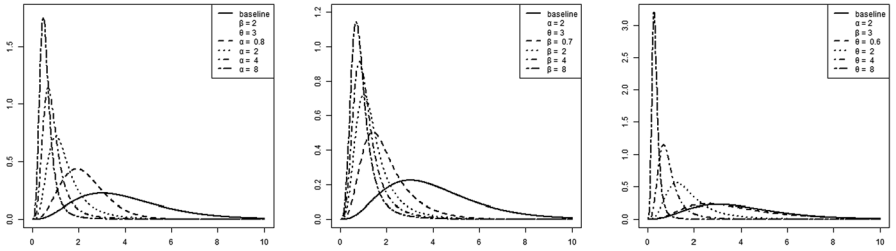

Fig. 2 Densities of the baseline distribution ($\Gamma(4, 1)$) and $G_{\alpha,\beta,\theta}$

where

$$T_{\alpha,\beta,\theta}(x) = \frac{(\alpha+x)^{\theta-1}(x+1)}{1+\beta\left((\alpha+x)^{\theta}-\alpha^{\theta}\right)}, \quad x \geq 0. \tag{8}$$

Hence, we may prove monotonicity properties for $h_{G_{\alpha,\beta,\theta}}$ looking at the monotonicity of $T_{\alpha,\beta,\theta}$, which will be addressed via $U_{\alpha,\beta,\theta}(x) = \frac{1}{T_{\alpha,\beta,\theta}(x)}$, for simplicity. After differentiation and some simple algebraic manipulation, one gets $U'_{\alpha,\beta,\theta}(x) = \frac{D_{\alpha,\beta,\theta}(x)}{(x+1)^2(\alpha+x)^{\theta}}$, where $D_{\alpha,\beta,\theta}(x) = \beta(1-\alpha)(\alpha+x)^{\theta} + (\beta\alpha^{\theta}-1)(\theta x+\alpha+\theta-1)$, and the sign of $U'_{\alpha,\beta,\theta}$ coincides with the sign of $D_{\alpha,\beta,\theta}$. We have that $D'_{\alpha,\beta,\theta}(x) = \theta\beta(1-\alpha)(\alpha+x)^{\theta-1} + \theta(\beta\alpha^{\theta}-1)$ and $D''_{\alpha,\beta,\theta}(x) = (1-\alpha)\beta\theta(\theta-1)(\alpha+x)^{\theta-2}$. Therefore, $D''_{\alpha,\beta,\theta}(x) \overset{\text{sgn}}{=} \text{sgn}((1-\alpha)(\theta-1))$, so $D'_{\alpha,\beta,\theta}$ is either increasing or decreasing. Now, the sign of $D_{\alpha,\beta,\theta}(0) = \alpha^{\theta}\beta\theta - \alpha - (\theta-1)$ will play a significant role.

Theorem 4.5 *Let $G_{\alpha,\beta,\theta}$ be given by (5) and $D_{\alpha,\beta,\theta}$ be the polynomial defined above.*

1. If $D_{\alpha,\beta,\theta}(0) < 0$, $(1-\alpha)(\theta-1) < 0$, and $F \in \text{IHR}$, then $G_{\alpha,\beta,\theta} \in \text{IHR}$.
2. If $D_{\alpha,\beta,\theta}(0) > 0$, $(1-\alpha)(\theta-1) > 0$, and $F \in \text{DHR}$, then $G_{\alpha,\beta,\theta} \in \text{DHR}$.
3. If $\alpha = 1$ or $\theta = 1$, $\beta \leq 1$ and $F \in \text{IHR}$, then $G_{\alpha,\beta,\theta} \in \text{IHR}$.
4. If $\alpha = 1$ or $\theta = 1$, $\beta \geq 1$ and $F \in \text{DHR}$, then $G_{\alpha,\beta,\theta} \in \text{DHR}$.

Proof In the first case, $D(x) < 0$ for every $x > 0$. Hence $U'_{\alpha,\beta,\theta}$ is always negative, so $U_{\alpha,\beta,\theta}$ is decreasing and, therefore, $T_{\alpha,\beta,\theta}(x) = \frac{1}{U_{\alpha,\beta,\theta}(x)}$ is increasing, so the conclusion is straightforward. The remaining cases are analogous, reversing signs and monotonicity directions for cases 2. and 4. □

Note that this result extends Theorem 3.3, allowing now for an interplay of the different parameters. Moreover, it is the presence of the parameter α that allows concluding about the monotonicity of the hazard rate for $\theta \neq 1$, which was out of reach in Theorem 3.3.

The preservation of the monotonicity of the odds ratio is easily described in analogous terms, extending the final part of Theorem 3.3.

Theorem 4.6 1. If $\theta \geq 1$ and $F \in \text{IOR}$, then $G_{\alpha,\beta,\theta} \in \text{IOR}$.
2. If $\theta \leq 1$ and $F \in \text{DOR}$, then $G_{\alpha,\beta,\theta} \in \text{DOR}$.

Proof Just note that $\lambda'_{G_{\alpha,\beta,\theta}}(x) \overset{\text{sgn}}{=} \lambda'_F(x)(\alpha+\Lambda_F(x)) + (\theta-1)\lambda_F^2(x)$, and the conclusion is immediate. □

4.2 Stochastic comparisons between $G_{\alpha,\beta,\theta}$ and F

We now address some stochastic ordering relations between the baseline distribution F and the family of transformed distributions $G_{\alpha,\beta,\theta}$ in the d-oMO model.

Theorem 4.7 1. If $\theta > 1$ and $\alpha^{\theta-1}\beta\theta > 1$, then $G_{\alpha,\beta,\theta} \leq_{st} F$.
2. If $\theta < 1$ and $\alpha^{\theta-1}\beta\theta < 1$, then $G_{\alpha,\beta,\theta} \geq_{st} F$.

Proof We need to characterise the sign of

$$\overline{G}_{\alpha,\beta,\theta}(x) - \overline{F}(x) = \frac{1}{1+\beta((\alpha+\Lambda_F(x))^\theta - \alpha^\theta)} - \frac{1}{\Lambda_F(x)+1} = H(\Lambda_F(x)),$$

where $H(x) = \frac{1}{1+\beta((\alpha+x)^\theta-\alpha^\theta)} - \frac{1}{x+1}$, which has the same sign variation as $P(x) = x - \beta(\alpha+x)^\theta + \alpha^\theta\beta$. After differentiation, we have $P'(x) = 1 - \theta\beta(\alpha+x)^{\theta-1}$ and $P''(x) = -\beta\theta(\theta-1)(\alpha+x)^{\theta-1}$. When $\theta > 1$, $P''(x) < 0$, so $P'(x)$ is decreasing. If $P'(0) = 1 - \beta\theta\alpha^{\theta-1} < 0$ it follows that $P'(x) < 0$, for every $x > 0$, hence $P(x)$ is decreasing. Since that $P(0) = 0$ we have the negativeness of $P(x)$. The case $\theta < 1$ is handled analogously. □

Sufficient conditions for the hazard rate order follow immediately by remarking that

$$H^*(x) = \frac{h_{G_{\alpha,\beta,\theta}}(x)}{h_F(x)} = \beta\theta T_{\alpha,\beta,\theta}(\Lambda_F(x)),$$

where $T_{\alpha,\beta.\theta}$ is defined by (8). Noting that $H^*(0) = \alpha^{\theta-1}$, taking into account the properties of $T_{\alpha,\beta.\theta}$ mentioned above, the following statement is obvious.

Theorem 4.8 1. If $\alpha^{\theta-1} > 1$ and $T_{\alpha,\beta,\theta}$ is increasing, then $G_{\alpha,\beta,\theta} \leq_{hr} F$.
2. If $\alpha^{\theta-1} < 1$ and $T_{\alpha,\beta,\theta}$ is decreasing, then $G_{\alpha,\beta,\theta} \geq_{hr} F$.

For a characterisation of the monotonicity of $T_{\alpha,\beta,\theta}$, please see the discussion preceding Theorem 4.5.

Now, the likelihood order follows easily.

Theorem 4.9 1. Assume that $\theta > 1$ and $F \leq_{hr} G_{\alpha,\beta,\theta}$. Then $G_{\alpha,\beta,\theta} \leq_{lr} F$.
2. Assume that $\theta < 1$ and $F \geq_{hr} G_{\alpha,\beta,\theta}$. Then $G_{\alpha,\beta,\theta} \geq_{lr} F$.

Proof Note that

$$\frac{g_{\alpha,\beta,\theta}(x)}{f(x)} = \frac{\beta\theta(\alpha+\Lambda_F(x))^\theta \overline{G}^2(x)}{\overline{F}(x)(\alpha\overline{F}(x)+F(x))} = \beta\theta(\alpha+\Lambda_F(x))^{\theta-1}\left(\frac{\overline{G}(x)}{\overline{F}(x)}\right)^2.$$

The monotonicity of the first parenthesis of the final expression on the right is fully defined by the sign of the exponent, while the monotonicity of the second term

depends on monotonicity the hazard rate order between the distributions F and $G_{\alpha,\beta,\theta}$. □

5 An enlarged log-logistic family of distributions

We have treated the oMO and d-oMO models, introduced in Definitions 3.1 and 4.1 by defining distributions through their odds functions, as distortions of some given underlying odds function Λ_F. Naturally, we may instead consider the new odds function as a distortion of the initial distribution function F. We mentioned, just before Definition 4.1, that the odds function for the PHR model corresponds to transforming F by the quantile function of a Pareto distribution. This approach may be extended to the full class of models considered in Definition 4.1, leading to the introduction of a new family of distributions.

Definition 5.1 The enlarged log-logistic distribution with parameters $\alpha \geq 0$, β, $\theta > 0$, denoted with $\mathrm{ELL}(\alpha, \beta, \theta)$ has distribution function

$$K_{\alpha,\beta,\theta}(x) = 1 - \frac{1}{\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}} + 1 - \alpha}, \quad x \geq 0. \tag{9}$$

The parameters β and $\frac{1}{\theta}$ are obviously scale and shape parameters, respectively, and α is a second shape parameter, having an effect on the asymmetry, skewness, and tail weight of the distribution. Moreover, it is straightforward to verify that $K_{0,1,1}$ is the standard log-logistic, already introduced before and denoted with $\mathcal{L}$, while $K_{0,\beta,\theta}$ represents the log-logistic with distribution function $\mathcal{L}_{\beta,\frac{1}{\theta}}(x) = 1 - \left((\frac{x}{\beta})^{\frac{1}{\theta}} + 1\right)^{-1}$. Moreover, the distributions $K_{1,\beta,\theta}$ correspond to the Pareto family.

Explicit expressions for the density $k_{\alpha,\beta,\theta}$, hazard rate $h_{\alpha,\beta,\theta}$, and quantile function $K^{-1}_{\alpha,\beta,\theta}$ for the distribution function $K_{\alpha,\beta,\theta}$ are given below:

$$k_{\alpha,\beta,\theta}(x) = \frac{\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}-1}}{\beta\theta\left(\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}} + 1 - \alpha\right)^2}, \qquad h_{\alpha,\beta,\theta}(x) = \frac{\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}-1}}{\beta\theta\left(\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}} + 1 - \alpha\right)}, \quad x \geq 0.$$

and

$$K^{-1}_{\alpha,\beta,\theta}(u) = \beta\left(\left(\frac{1}{1-u} + \alpha - 1\right)^\theta - \alpha^\theta\right), \quad u \in [0,1].$$

An illustration of the effect of the parameters is shown in Fig. 3, describing the behaviour of the density function. It is clear that mass concentrates near the origin, with a shift to the right for $\alpha < 1$ and $\theta < 1$. Moreover, due to the exponent of the

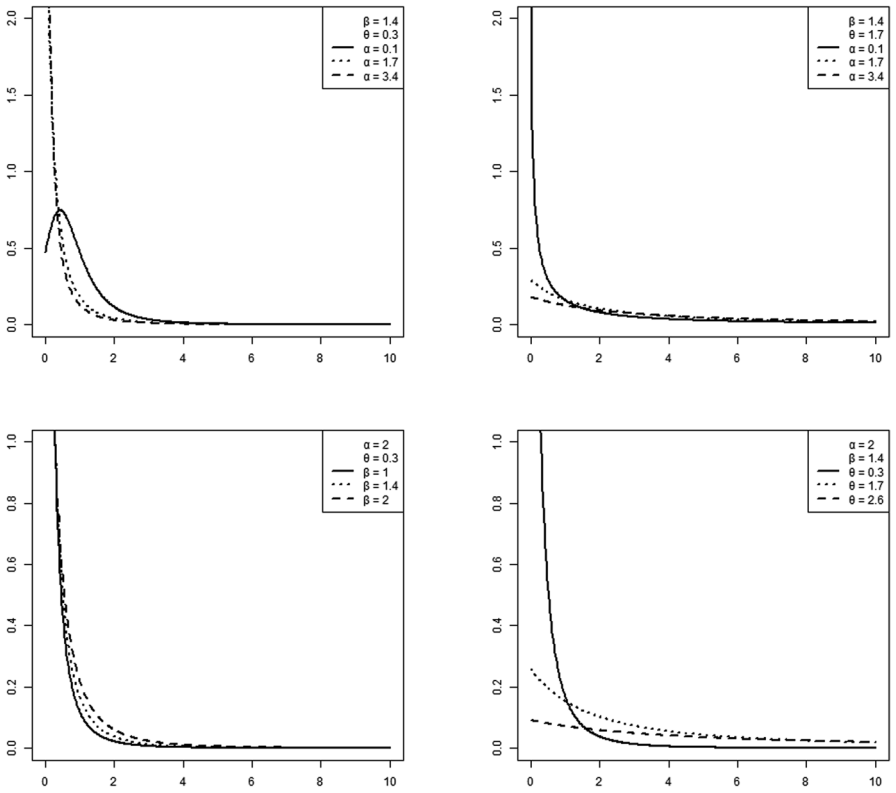

Fig. 3 Some plots for the density of $K_{\alpha,\beta,\theta}$

form $\frac{1}{\theta} - 1$, the concentration of mass near the origin varies in opposite directions with respect to the parameters α or β, depending on whether $\theta < 1$ or $\theta > 1$. On the other hand, increasing θ produces heavier tails.

It is now evident that $\Lambda_{G_{\alpha,\beta,\theta}}(x) = K^{-1}_{\alpha,\beta,\theta} \circ F(x)$, highlighting the role of $K_{\alpha,\beta,\theta}$ in modelling the distortion of the odds function. Therefore, ordering properties within the $K_{\alpha,\beta,\theta}$ family translate easily into the distributions in the d-oMO model, the convexity of the odds of $G_{\alpha,\beta,\theta}$ being the most obvious, corresponding to the convex transform order between $K_{\alpha,\beta,\theta}$ and F. In other words, the convexity properties of the baseline distribution F with respect to $K_{\alpha,\beta,\theta}$ are inherited by $G_{\alpha,\beta,\theta}$. Recall that the convexity of the odds function defines interesting classes, namely the IOR and DOR families of distributions (see Lando et al. 2022). This naturally leads to an interest in exploring stochastic ordering relationships within the family of distributions defined by (9).

The increasingness of the hazard rate or the odds rate is simple to characterise, as described next.

Theorem 5.2 1. If $\alpha + \theta > 1$ then $K_{\alpha,\beta,\theta} \in$ DHR, for every $\beta > 0$.
2. If $\theta \leq 1$ then $K_{\alpha,\beta,\theta} \in$ IOR, while if $\theta \geq 1$ then $K_{\alpha,\beta,\theta} \in$ DOR.

Proof For part 1., note that

$$h'_{\alpha,\beta,\theta}(x) = -\left(\theta\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}} + (1-\alpha)(\theta-1)\right).$$

Since $h'_{\alpha,\beta,\theta}(0) = -(\alpha+\theta-1)$, it follows that $h'_{\alpha,\beta,\theta}(x) < 0$ for every $x \geq 0$, given that h' is decreasing. For part 2., the odds rate of $K_{\alpha,\beta,\theta}$ is given by $\lambda_{K_{\alpha,\beta,\theta}}(x) = \frac{1}{\beta\theta}\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}-1}$, so the conclusion is obvious. □

The following result characterises the $\leq_{st}$-order comparability within the ELL family.

Theorem 5.3 *Assume the parameters $\alpha \geq 0$, $\beta, \theta > 0$ and $\alpha_1 \geq 0$, $\beta_1, \theta_1 > 0$ of the enlarged log-logistic distribution functions (9) satisfy one of the following assumptions:*

(ST1) $\theta < \theta_1$, $\alpha^{\theta-1}\beta\theta < \alpha_1^{\theta_1-1}\beta_1\theta_1$ and $\alpha_1(1-\theta) - \alpha(1-\theta_1) \geq 0$;
(ST2) $\theta = \theta_1$, $\beta < \beta_1$, $\alpha^{\theta-1}\beta < \alpha_1^{\theta_1-1}\beta_1$ and $(1-\theta)\left(\alpha_1^{\theta}\beta_1 - \alpha^{\theta}\beta\right) > 0$.

Then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha_1,\beta_1,\theta_1}$.

Proof For the general set of parameters (α, β, θ) denote $\overline{K}_{\alpha,\beta,\theta}(x) = 1 - K_{\alpha,\beta,\theta}(x)$, and define

$$V(x) = \frac{1}{\overline{K}_{\alpha,\beta,\theta}(x)} - \frac{1}{\overline{K}_{\alpha_1,\beta_1,\theta_1}(x)} = \left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}} - \left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}} + \alpha_1 - \alpha.$$

Noting that $\overline{K}_{\alpha_1,\beta_1,\theta_1}(x) - \overline{K}_{\alpha,\beta,\theta}(x) \overset{\text{sgn}}{=} V(x)$, the proof is concluded if we prove that $V(x) \geq 0$, for every $x \geq 0$. It is obvious that $V(0) = 0$. We separate the two cases, according to which assumption is satisfied.

(ST1): We have $V(+\infty) = \infty \times \text{sgn}\left(\frac{1}{\theta} - \frac{1}{\theta_1}\right) = +\infty$. Differentiating, we find

$$V'(x) = \frac{1}{\beta\theta}\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}-1} - \frac{1}{\beta_1\theta_1}\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}-1},$$

so $V'(0) = \frac{\alpha^{1-\theta}}{\beta\theta} - \frac{\alpha_1^{1-\theta_1}}{\beta_1\theta_1} > 0$. Now, if we prove that $V'(x) \geq 0$, for every $x \geq 0$, it follows that V is increasing, hence $V(x) \geq 0$, and the conclusion follows. Therefore, we need to prove that

$$V'(x) \geq 0 \quad \Leftrightarrow \quad P(x) = \frac{\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}-1}}{\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}-1}} \geq \frac{\beta\theta}{\beta_1\theta_1}.$$

Noting that $P(0) = \frac{\alpha^{1-\theta}}{\alpha_1^{1-\theta_1}} > \frac{\beta\theta}{\beta_1\theta_1}$ and $P(+\infty) = +\infty$, we now look at the monotonicity of P. Differentiating, one observes that $P'(x) \overset{\text{sgn}}{=} L(x)$, where $L(x) = \frac{1}{\beta\beta_1}\left(\frac{1}{\theta} - \frac{1}{\theta_1}\right)x + \frac{1-\theta}{\theta}\frac{\alpha_1^{\theta_1}}{\beta} - \frac{1-\theta_1}{\theta_1}\frac{\alpha^{\theta}}{\beta_1}$. The assumptions imply that both the slope and intercept of $L(x)$ are positive, hence P is increasing, implying that $P(x) \geq \frac{\beta\theta}{\beta_1\theta_1}$, thus $V'(x) = 0$ has no solution.

(ST2): This case is treated analogously, so we just highlight the relevant differences. We now have $V(+\infty) = \infty \times \text{sgn}\left(\frac{1}{\beta} - \frac{1}{\beta_1}\right) = +\infty$, and $P'(x) \overset{\text{sgn}}{=} (1-\theta)\left(\alpha_1^{\theta}\beta_1 - \alpha^{\theta}\beta\right)$, assumed to be positive. □

The previous result allows for an immediate pointwise comparison of the odds ratio of the $G_{\alpha,\beta,\theta}$ family.

Corollary 5.4 *Let $G_{\alpha,\beta,\theta}$ be given by (5). Under either of the assumptions of Theorem 5.3, it holds that $\Lambda_{G_{\alpha,\beta,\theta}}(x) \leq \Lambda_{G_{\alpha_1,\beta_1,\theta_1}}(x)$ for every $x \geq 0$.*

Proof Remember that $\Lambda_{G_{\alpha,\beta,\theta}}(x) = K^{-1}_{\alpha,\beta,\theta}(F(x))$. Under the assumptions of Theorem 5.3, we have that $K_{\alpha,\beta,\theta}(x) \geq K_{\alpha_1,\beta_1,\theta_1}(x)$, for every $x \geq 0$. But this is equivalent to $K^{-1}_{\alpha,\beta,\theta}(x) \leq K^{-1}_{\alpha_1,\beta_1,\theta_1}(x)$ for every $x \geq 0$, so the result follows immediately. □

Recalling the characterisation of the $\leq_{st}$-order using the odds function (see Proposition 2.3), the previous result can be rewritten as follows, describing conditions for the usual stochastic order within the distributions of the d-oMO model.

Corollary 5.5 *Let $G_{\alpha,\beta,\theta}$ be given by (5). Under either of the assumptions of Theorem 5.3, it holds that $G_{\alpha_1,\beta_1,\theta_1} \leq_{st} G_{\alpha,\beta,\theta}$.*

Conditions for the particular case of one parameter comparison, corresponding to the ELL that characterise the PHR, the proportional odds or the proportional log-odds models, are immediate from Theorem 5.3. We state the result, for sake of completeness.

Corollary 5.6 *For the enlarged log-logistic distribution functions (9) we have that:*

1. If $\alpha \geq \alpha_1 \geq 0$ and $\theta \leq 1$ then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha_1,\beta,\theta}$ for every $\beta > 0$.
2. If $\beta \leq \beta_1$ and $\theta \leq 1$ then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha,\beta_1,\theta}$ for every $\alpha \geq 0$.
3. If $\theta < \theta_1 \leq 1$, $\alpha^{\theta_1-\theta} > \frac{\theta}{\theta_1}$ and $\frac{1-\theta}{\alpha^{\theta}\theta} > \frac{1-\theta_1}{\alpha^{\theta_1}\theta_1}$ then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha,\beta,\theta_1}$ for every $\beta > 0$.

We now prove a general set of conditions providing the $\leq_{hr}$-comparability within the ELL family.

Theorem 5.7 *Assume the parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$, $\theta_1 > 0$ satisfy the following assumptions:*

(HR1) *(i)* $\beta\theta\alpha^{\theta-1} \leq \beta_1\theta_1\alpha_1^{\theta_1-1}$, and *(ii)* $\beta\theta\alpha^{\theta} \leq \beta_1\theta_1\alpha_1^{\theta_1}$,

(HR2) $\theta < \theta_1$,

(HR3) $(1-\alpha_1)(\theta_1-1) \geq 0$,

(HR4) $\left(\frac{1}{\alpha}-1\right)(\theta-1) \leq \left(\frac{1}{\alpha_1}-1\right)(\theta_1-1)$.

Then $K_{\alpha,\beta,\theta} \leq_{hr} K_{\alpha_1,\beta_1,\theta_1}$.

Proof We shall prove that

$$\begin{aligned} V(x) =& \frac{1}{h_{\alpha_1,\beta_1,\theta_1}(x)} - \frac{1}{h_{\alpha,\beta,\theta}(x)} \\ =& \beta_1\theta_1\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right) + (1-\alpha_1)\beta_1\theta_1\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{1-\frac{1}{\theta_1}} \\ & - \beta\theta\left(\frac{x}{\beta}+\alpha^{\theta}\right) - (1-\alpha)\beta\theta\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{1-\frac{1}{\theta}} \geq 0, \end{aligned}$$

which clearly implies the conclusion. We start by noting that, taking into account *(HR1-i)* and *(HR2)*, $V(0) = \beta_1\theta_1\alpha_1^{\theta_1-1} - \beta\theta\alpha^{\theta-1} \geq 0$ and $V(+\infty) = \infty \times \operatorname{sgn}(\theta_1-\theta) = +\infty$, hence the conclusion follows if we prove that V is increasing. Direct differentiation gives

$$V'(x) = \theta_1 - \theta + (1-\alpha_1)(\theta_1-1)\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{-\frac{1}{\theta_1}} - (1-\alpha)(\theta-1)\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{-\frac{1}{\theta}}, \quad (10)$$

so, given *(HR2)* and *(HR3)*, the nonnegativity of V' follows if we prove that

$$Q(x) = \frac{\beta_1^{\frac{1}{\theta_1}}}{\beta^{\frac{1}{\theta}}} \frac{\left(x+\beta\alpha^{\theta}\right)^{\frac{1}{\theta}}}{\left(x+\beta_1\alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}}} \geq \frac{(1-\alpha)(\theta-1)}{(1-\alpha_1)(\theta_1-1)}.$$

It is easily seen that *(HR3)* and *(HR1-ii)* imply that $Q'(x) \geq 0$ for every $x \geq 0$, hence Q is increasing. Finally, *(HR4)* means that $Q(0) \geq \frac{(1-\alpha)(\theta-1)}{(1-\alpha_1)(\theta_1-1)}$, so the theorem is proved. □

Theorem 5.7 does not allow to choose $\alpha = 0$, therefore leaving out of the comparisons the important case of the log-logistic distribution $\mathcal{L}$, as the expression (10) means, when taking $x = 0$, that α appears as a denominator. The way out of this can be sorted adapting the expressions above by continuity when $\alpha \to 0$.

Corollary 5.8 *Assume the parameters $\beta > 0$, $0 < \theta \leq 1$ and $\alpha_1 > 0$, $\beta_1 > 0$, $\theta_1 > 0$ satisfy (HR2) and (HR3). Then $K_{0,\beta,\theta} \leq_{hr} K_{\alpha_1,\beta_1,\theta_1}$.*

Proof With respect to the proof of Theorem 5.7 note that, after allowing $\alpha \to 0$, we need that $\theta \leq 1$ to fulfill the appropriate version of $Q(0) = 0 \geq \frac{\theta-1}{(1-\alpha_1)(\theta_1-1)}$. □

Moreover, note that Theorem 5.7 proof's argument depends crucially on $\theta < \theta_1$, and breaks down if we assume equality of these two parameters.

Corollary 5.9 *Assume the parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$ are such that $\alpha > \alpha_1$, $\beta\alpha^{\theta-1} \leq \beta_1\alpha_1^{\theta-1}$ and*

$$\begin{cases} if \theta \geq 1, \quad (1-\alpha)\beta^{\frac{1}{\theta}} \leq (1-\alpha_1)\beta_1^{\frac{1}{\theta}}, \\ if \theta \leq 1, \quad \beta\alpha^{\theta} < \beta_1\alpha_1^{\theta}. \end{cases}$$

Then $K_{\alpha,\beta,\theta} \leq_{hr} K_{\alpha_1,\beta_1,\theta}$.

Proof Rewrite the function

$$V(x) = \frac{1}{h_{\alpha_1,\beta_1,\theta}(x)} - \frac{1}{h_{\alpha,\beta,\theta}(x)}$$
$$= \left(\beta_1\alpha_1^{\theta} - \beta\alpha^{\theta}\right)\theta + (1-\alpha_1)\beta_1\theta\left(\frac{x}{\beta_1} + \alpha_1^{\theta}\right)^{1-\frac{1}{\theta}} - (1-\alpha)\beta\theta\left(\frac{x}{\beta} + \alpha^{\theta}\right)^{1-\frac{1}{\theta}}.$$

The assumptions imply that $V(0) \geq 0$ and $V(+\infty) \geq 0$, possibly equal to $+\infty$. The equation $V'(x) = 0$ translates into

$$1 + \frac{\beta\alpha^{\theta} - \beta_1\alpha_1^{\theta}}{x + \beta_1\alpha_1^{\theta}} = \frac{\beta}{\beta_1}\left(\frac{1-\alpha}{1-\alpha_1}\right)^{\theta},$$

which may have at most one root for $x \geq 0$. Moreover, $V'(0) = \frac{\alpha-\alpha_1}{\alpha\alpha_1} > 0$, therefore V starts increasing at $x = 0$. Hence, $V(x) > 0$ for every $x \geq 0$, and the conclusion follows. □

Again, as for the general result, Corollary 5.9 does not include the case $\alpha = 0$, but this can be handled in exactly the same way as in Corollary 5.8.

Corollary 5.10 *Assume the parameters $\beta > 0$, $\theta > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$ are such that $\beta^{\frac{1}{\theta}} \leq (1-\alpha_1)\beta_1^{\frac{1}{\theta}}$. Assume, further, than one of the following conditions is satisfied:*

(HR5) $(1-\alpha_1)(\theta-1) \geq 0$;

(HR6) $(1-\alpha_1)(\theta-1) < 0$ and $\alpha_1 + (1-\alpha_1)\beta_1^{1-\frac{1}{\theta}} \frac{(1-\alpha_1^{\theta})^{1-\frac{1}{\theta}} - \beta}{\left(\beta_1(1-\alpha_1^{\theta}) - \beta\right)^{1-\frac{1}{\theta}}} \geq 0$.

Then $K_{0,\beta,\theta} \leq_{hr} K_{\alpha_1,\beta_1,\theta}$.

Proof We need to look now at the sign of

$$V(x)=\frac{1}{h_{\alpha_1,\beta_1,\theta}(x)}-\frac{1}{h_{0,\beta,\theta}(x)}=\beta_1\alpha_1^{\theta}\theta+(1-\alpha_1)\beta_1\theta\left(\frac{x}{\beta_1}+\alpha_1^{\theta}\right)^{1-\frac{1}{\theta}}-\beta\theta\left(\frac{x}{\beta}\right)^{1-\frac{1}{\theta}}.$$

We have $V(0)=\beta_1\alpha_1^{\theta-1}\theta>0$. Moreover,

$$V(+\infty)=\begin{cases}\infty\times\operatorname{sgn}\left((1-\alpha_1)\beta_1^{\frac{1}{\theta}}-\beta^{\frac{1}{\theta}}\right) & \text{if } 1-\dfrac{1}{\theta}>0,\\ \beta_1\alpha_1^{\theta} & \text{if } 1-\dfrac{1}{\theta}<0.\end{cases}$$

Therefore, under our assumptions, $V(+\infty)=+\infty$ for every $\theta>0$. Seeking for extreme points of V, we need to solve $V'(x)=0$, which translates into $P(x)=\frac{x}{x+\beta_1\alpha_1^{\theta}}=\frac{\beta}{\beta_1}\frac{1}{(1-\alpha_1)^{\theta}}$. It is easy to verify that P is increasing, $P(0)=0$, $P(+\infty)=1$, and the right hand side of he equation is less or equal than 1, so this equation has exactly one solution, equal to $x_0=\frac{\beta\beta_a\alpha_1^{\theta}}{\beta_1(1-\alpha_1^{\theta})-\beta}$. Assuming *(HR5)*, it follows that $V'(x)\geq 0$, for every $x\geq 0$, hence V remains positive. If assuming *(HR6)*, V has a minimum at x_0, and our assumptions mean that $V(x_0)\geq 0$ so, again, we conclude that V stays positive, thus concluding the proof. □

Finally, a characterisation of convex transform order relationships.

Theorem 5.11 *For the enlarged log-logistic distribution functions (9) we have that:*

1. If $\theta\leq\theta_1$ and $\alpha(\theta_1-1)+\alpha_1(1-\theta)\geq 0$, then for every $\beta,\beta_1>0$, $K_{\alpha,\beta,\theta}\leq_c K_{\alpha_1,\beta_1,\theta_1}$.
2. If $\theta\geq\theta_1$ and $\alpha(\theta_1-1)+\alpha_1(1-\theta)\leq 0$, then for every $\beta,\beta_1>0$, $K_{\alpha_1,\beta_1,\theta_1}\leq_c K_{\alpha,\beta,\theta}$.

Proof First note that as the β is a scale parameter and the convex transform order is invariant with respect to scale parameters, we may assume that $\beta=\beta_1=1$. We need to look at the convexity/concavity of

$$\psi(x)=K^{-1}_{\alpha_1,1,\theta_1}\circ K_{\alpha,1,\theta}(x)=\left(\left(x+\alpha^{\theta}\right)^{\frac{1}{\theta}}+\alpha_1-\alpha\right)^{\theta_1}-\alpha_1^{\theta_1}.$$

Simple differentiation and simplification show that $\psi''(x)\overset{\operatorname{sgn}}{=}(\theta_1-\theta)\left(x+\alpha^{\theta}\right)^{\frac{1}{\theta}}+(1-\theta)(\alpha_1-\alpha)$. Therefore, ψ is convex if $\theta_1-\theta\geq 0$ and $\psi''(0)=\alpha(\theta_1-1)+\alpha_1(1-\theta)>0$, and it is concave if both these two inequalities are reversed. □

The following particular cases are now obvious.

Corollary 5.12 *For the enlarged log-logistic distribution functions (9) we have that:*

1. If $\theta \geq 1$, then for every $\alpha \geq 0$, $\mathcal{L} = K_{0,\beta,1} \leq_c K_{\alpha,\beta,\theta} \leq_c K_{0,\beta,\theta}$.
2. If $\theta \leq 1$, then for every $\alpha \geq 0$, $K_{0,\beta,\theta} \leq_c K_{\alpha,\beta,\theta} \leq_c K_{0,\beta,1} = \mathcal{L}$.

Remark 5.13 As mentioned above, the IOR family may be characterised as the class of distributions that are dominated, with respect to the convex transform order, by the standard log-logistic $K_{0,1,1}$(which is equivalent, for this purpose, to $K_{0,\beta,1}$, for every $\beta > 0$). Denote with $D_{\alpha,\beta,\theta}$ the family of distributions that are dominated, with respect to the convex transform order, by the $K_{\alpha,\beta,\theta}$ distribution. Then, we have that $\mathrm{IOR} = D_{0,\beta,1}$, for every $\beta > 0$. Moreover, the transitivity of the $\leq_c$-ordering implies that, for $\theta \geq 1$ and $\alpha \geq 0$, $\mathrm{IOR} = D_{0,\beta,1} \subset D_{\alpha,\beta,\theta} \subset D_{0,\beta,\theta}$. This inclusion implies that, for this choice of parameters, the IOR class remains nested within this more general family, hence meaning that the requirement that $G \in D_{\alpha,\beta,\theta}$ is less stringent that $G \in \mathrm{IOR}$. As emphasized in Lando et al. (2022), the IOR already encompasses several well-known distributions with interesting shape properties, namely, allows heavy tailed distributions or for bathtub shaped hazard rates.

In Theorem 4.6, we described conditions implying the monotonicity of the odds rate $\lambda_{G_{\alpha,\beta,\theta}}$. This monotonicity, following the Lando et al. (2022), translates into either $G_{\alpha,\beta,\theta} \leq_c \mathcal{L} = K_{0,1,1}$, equivalent to $G_{\alpha,\beta,\theta} \in \mathrm{IOR}$, or $\mathcal{L} = K_{0,1,1} \leq_c G_{\alpha,\beta,\theta}$, equivalent to $G_{\alpha,\beta,\theta} \in \mathrm{DOR}$. We may now describe a more general form of the convex transform relations between the $G_{\alpha,\beta,\theta}$ and $K_{\alpha,\beta,\theta}$ families of distributions.

Theorem 5.14 *Let $G_{\alpha,\beta,\theta}$ be described by (5) and $K_{\alpha_1,\beta_1,\theta_1}$ as in (9). If $F \in \mathrm{IOR}$ and $\theta, \theta_1 \geq 1$, then $G_{\alpha,\beta,\theta} \leq_c K_{\alpha_1,\beta_1,\theta_1}$. On the other hand, if $F \in \mathrm{DOR}$ and $\theta, \theta_1 \leq 1$, then $K_{\alpha_1,\beta_1,\theta_1} \leq_c G_{\alpha,\beta,\theta}$.*

Proof Assume that $F \in \mathrm{IOR}$ and $\theta, \theta_1 \geq 1$. Due to the invariance of the convex transform order with respect to scale parameters, we may assume $\beta_1 = 1$. Hence, we want to prove the convexity of

$$\psi(x) = K^{-1}_{\alpha_1,1,\theta_1} \circ G_{\alpha,\beta,\theta}(x) = \left(\beta\left((\alpha + \Lambda_F(x))^{\theta} - \alpha^{\theta}\right) + \alpha_1\right)^{\theta_1} - \alpha_1^{\theta_1}.$$

Differentiation shows that

$$\psi'(x) = \beta\theta\theta_1\left(\beta\left((\alpha + \Lambda_F(x))^{\theta} - \alpha^{\theta}\right) + \alpha_1\right)^{\theta_1 - 1} \lambda_F(x)\,(\alpha + \Lambda_F(x))^{\theta - 1},$$

which, under our assumptions, is clearly increasing, so ψ is convex. The second statement is proved analogously. □

Theorem 5.15 *Let $G_{\alpha,\beta,\theta}$ be described by (5) and $K_{\alpha_1,\beta_1,\theta_1}$ as in (9). If $F \in \mathrm{IOR}, \theta, \theta_1 \geq 1$ and $\beta\beta_1\theta\theta_1 f(0)\alpha^{\theta-1}\alpha^{\theta_1-1} \geq 1$, then $G_{\alpha,\beta,\theta} \leq_{disp} K_{\alpha_1,\beta_1,\theta_1}$. On the other hand, if $F \in \mathrm{DOR}$, $\theta, \theta_1 \leq 1$ and $\beta\beta_1\theta\theta_1 f(0)\alpha^{\theta-1}\alpha^{\theta_1-1} \leq 1$, then $K_{\alpha_1,\beta_1,\theta_1} \leq_{disp} G_{\alpha,\beta,\theta}$.*

Proof The result follows by studying the monotonicity of the function $\phi(x) = K^{-1}_{\alpha_1,\beta_1,\theta_1} \circ G_{\alpha,\beta,\theta}(x) - x$. Observe that if $F \in \mathrm{IOR}$ and $\theta, \theta_1 \geq 1$, ϕ' is increasing while the additional assumption ensures that $\phi'(0) \geq 0$, establishing the nonnegativeness of ϕ'. The second part of the theorem follows in a similar manner. □

Remark 5.16 Notice that $K_{\alpha_1,\beta_1,\theta_1}(0) = G_{\alpha,\beta,\theta}(0) = 0$. Thus, under the same conditions as in Theorem 5.15 we can easily get the respective results for the usual stochastic order by applying Theorem 3.B.13(a) of Shaked and Shanthikumar (2007).

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Declarations

Conflict of interest The authors declare no competing interests.

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