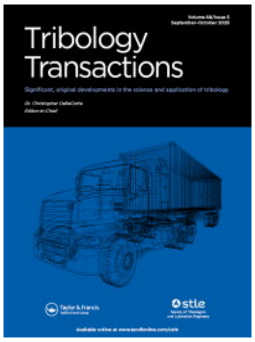

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Estimating Mixed & Boundary Friction from the Lambda Ratio: Incorporating the Effect of Surface Character

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ABSTRACT

The authors have previously reported a method to estimate boundary and mixed friction coefficients using lambda ratio values and friction data from mini traction machines (MTMs). This article reports a more general approach that accounts for the surface height distribution of the topographies at the interface by employing the overlap functions of idealized probability height distributions of the interface components. The overlap coefficients of probability density functions relevant to engineering rough surfaces have been calculated, and simple analytical expressions have been obtained for exponential, Gaussian, and Rayleigh distribution functions. This is believed to be the first time overlap coefficients of probability density functions have been calculated and applied in a tribological context. It has been found that the derived expressions can be used to predict mixed/boundary friction for rough surface lubricated contacts and are in good agreement with published experimental data. It has also been noted that rough surfaces with identical Gaussian distribution functions take an especially simple form and that surfaces that are described by the Rayleigh probability density function provide a good model for "run-in" surfaces. This feature allows tribologists to make quick and simple comparisons of the friction of new and worn, or run-in, machine components.

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Mixed lubrication; boundary lubrication; surface roughness; friction

Introduction

As the focus on reducing machine energy losses increases, there is a trend toward using lower lubricant viscosities, decreased lubricant sump volumes, and increased loads and temperatures to lower the impact of shear-related losses. This results in an increased likelihood of machines operating in the mixed and boundary lubrication regimes. Consequently, there is increasing interest in the accurate prediction of mixed and boundary friction to enable accurate modeling of the design and operation of modern machine components as they transition into new operating conditions and move from traditional full fluid film (hydrodynamic) operating conditions.

Previous work by the authors has estimated boundary and mixed friction coefficients for contacts lubricated with both non-additivated lubricants (*1*) and lubricants with friction modifiers and antiwear additives. (*2*) The technique used the values of the lambda ratio and "run-in" friction data from mini traction machines and was effective in providing a first approximation for friction coefficients in many applications. However, its accuracy and general application to a wider range of surfaces with different topography were restricted by the surface character of the data sources. This article reports a more general approach that accounts for the surface height distribution of the topographies at the interface. It is applicable for both new surfaces and the surfaces of run-in contacts. The method employs the overlap functions of idealized probability height distributions of the interface components.

Mixed and boundary friction results from the interaction of rough surface asperity contacts, in situations where the lubricant film is not sufficiently thick to prevent metal-to-metal contact. In this situation, the average lubricating film thickness and the probability density function of asperity surface heights of the interface components are important in determining the amount of surface asperity contact. In this article the authors explore the idea of using simple geometric parameters to characterize the "level of surface interaction" that leads to specific levels of friction.

The most well-known geometric character that is commonly used at present to define the level of interaction of two surfaces and the magnitude of friction at a lubricated interface is the lambda ratio (λ). It is given by the ratio of the distance between the mean lines of the surfaces (h) (considered to be the "lubricating film thickness") and the combined root mean square (RMS) roughness of each surface σ_1 and σ_2:

$$\lambda = \frac{h}{\sqrt{{\sigma_1}^2 + {\sigma_2}^2}} \qquad [1]$$

It is commonly accepted by most tribologists that lubrication regimes are defined by the values of lambda, so when λ

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< 1 the contact is in the boundary lubrication regime, when $1 < \lambda < \lambda_c$, (where λ_c is usually taken to be 3 or 4) the contact is in the mixed lubrication regime, and if $\lambda > \lambda_c$ the contact is considered to be in fluid film lubrication with surfaces completely (or almost completely) separated.

The lambda ratio has been widely used by tribologists for many years, but it is unclear who first developed or applied it. It is reasonable to assume that it is based on the underlying principle that if the two surfaces have Gaussian probability density functions for the asperity peak heights, different levels of overlap, assessed by h and the combined RMS roughness, can be used to "define" contact regimes. The basis of this is most likely founded in the properties of the Gaussian distribution. The area under the distribution within one, two, and three standard deviations of the mean encompasses approximately 68.3%, 95.5%, and 99.7%, respectively, of the distribution area, and it is reasonable to assign the different lubrication regimes to these values (in a one-sided fashion).

Over many decades several researchers have tried to relate the values of the lambda ratio to friction coefficients in the mixed and boundary regimes by various means, with differing levels of success. (*3–8*)

However, given the well-established use of the lambda ratio, it becomes easy to overlook the fact that it is based around one distribution form, so when other surface distributions are used in experiments or models, to predict friction coefficients from lambda values, differences will arise.

Surfaces with differing height distributions, reflecting the character of different finishing operations, can have significantly different profile features while having the same statistical parameterization. (*9*) Consequently, when using the lambda ratio, it is not possible to assign a unique value of this parameter that reflects the transitions between lubrication regimes for contacting surfaces for all forms of topographic character as is conventionally accepted.

It may be controversial to assume that a simple geometric factor can be used to predict friction as a proxy for a range of complex physical interactions involving shear of thin lubricating films, interlocking of asperities, elastic and plastic deformation of asperities, and other effects arising during relative motion under an applied normal contact load. However, this principle is already established by the lambda ratio, and the work presented here is intended to extend the range of validity of the general idea by developing equations to more correctly address contact between surfaces with differing height distributions.

There are various established surface contact models that have been employed in approaches to the prediction of mixed/boundary friction. They include those that assume elastic deformation of asperities, (*10–14*) and others that assume a degree of plastic deformation of asperities. (*15–18*) In addition, recent experimental data are available (*19*) and useful fitted engineering approximations for the estimation of mixed/boundary friction have been reported based on these data. (*1*) Historically, Bowden and Tabor (*20*) assumed that contacting asperities would deform and flow plastically, so that real contact area would increase until the contact pressure dropped below the material yield stress. In their model, the highest asperities would be wiped away, and the contact pressure would be constant and equal to the material yield stress. However, tribologists at the time could not understand why machine elements lasted so long if rough surface interaction was always plastic. It was then realized that although plastic deformation and adhesive wear of asperities do indeed occur when brand new surfaces slide over each other, it is the highest asperities that will contact first, and these will be the first to be worn off by an adhesive wear mechanism; over time, the rough surface becomes smoother, and all the highest asperities will be worn off. Once this "running-in" process (*21*) has occurred, contact of the rough surfaces will be over a much larger number of smaller asperities, the average contact pressure on each asperity will be greatly reduced (compared to that acting on the highest asperities of brand new surfaces), and much of the subsequent deformation of asperities will be by elastic deformation, (*10–14*) which helps to explain the longevity of machine elements. Greenwood and Williamson (*11*) and then Greenwood and Tripp (*12*) introduced widely used models for the elastic contact of rough surfaces, and the Greenwood and Tripp model (*12*) is still very widely used today when predicting the real area of contact in order to predict the amount of mixed lubrication in a contact. However, recent research (*22*) suggests that the Greenwood and Tripp model (*12*) significantly underestimates the amount of mixed friction in a contact. The work reported here attempts to find simple models that can be used to estimate the proportion of mixed/boundary friction in a contact, X, as a function of the well-established lambda ratio, and that are also consistent with available experimental data. The key aim is to capture how the influence of differences in surface character may lead to slightly differing friction curves.

The work takes a detailed look at surface height probability density functions, to understand the impact of interface topography on asperity interaction and boundary and mixed friction. In particular, the overlap coefficient of different probability density functions is analytically evaluated for some of the common types of distribution expected for engineering rough surfaces, such as exponential, Gaussian, and Rayleigh distributions. It is found that when there are two probability density functions for which the centers are separated by a certain distance, the overlap coefficient is a measure of the "overlap" of the tails of the two distributions. The overlap coefficient can be calculated, as a function of the distance of separation of the centers, and it is shown that the normalized overlap coefficient provides a reasonable estimate of mixed/boundary friction in a contact compared to published experimental data.

The overlap coefficient of rough surface probability density functions

In order to fully understand the behavior of surfaces in contact, it is desirable to use three-dimensional models that describe surface shape in x, y, and z directions, incorporating both lateral and vertical characteristics of the topography. While this is possible for contact simulations, three-dimensional (3-D) data are less commonly used in analytical modeling. In the analytical work presented here, the authors have opted to reduce the complexity of the problem by adopting a simplified (two-dimensional, 2-D) description of a surface that incorporates only the height characteristics of the surfaces in contact in the form of the surface height distribution.

Conventionally, rough surface asperity heights can be described using a probability density function. (*23*,*24*) Greenwood and Williamson have reported the use of both exponential and Gaussian distributions in their work. (*11*) These probability density functions (suitably normalized) can be written as follows.

Exponential distribution:

$$P(z) = \frac{1}{2\sigma} \exp\left(-\frac{|z-\mu|}{\sigma}\right) \quad [2]$$

Gaussian distribution:

$$P(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right) \quad [3]$$

In these two equations, $P(z)$ is the probability density function, μ is the mean of the distribution, and σ is the standard deviation of the distribution.

One advantage of using this approach to predict friction is that rough surface probability density functions (height distributions) can be measured experimentally using various systems (see for example (*22*)), so surface data can be collected easily for the purposes of friction prediction.

New surfaces produced by a range of finishing process such as grinding can often be reasonably well approximated by Gaussian height distributions.

The height distribution of many "run-in" surfaces where the highest surface asperities have been worn off is an important category to understand and represent. A probability distribution that can usefully be used as a model for such "run-in" surfaces is the Rayleigh distribution:

$$P(z) = \frac{z}{\sigma^2} \exp\left(-\frac{z^2}{2\sigma^2}\right) \quad [4]$$

This distribution function is only defined for $z > 0$, and the peak of the distribution occurs when $z = \sigma$.

In statistics, the overlap coefficient of two probability density functions provides an estimate of the similarity of two such functions. (*25*) For the case of two identical Gaussian density functions, Fig. 1 shows how the overlap coefficient is defined. The area of overlap of the "tails" of the distributions is defined to be $C(d)$, and the normalized overlap coefficient (OVC) is equal to $C(d)/C(0)$. In Fig. 1, the two distributions are identical normalized Gaussian probability density functions, so in this case, $C(0) = 1$, but this will generally not be the case for different probability density functions.

For identical Gaussian probability density functions, $P(z)$ (one with center at $z=0$) and $P(z)$ (with center at $z=d$), the intersection will occur at $z=d/2$, and it is then possible to write

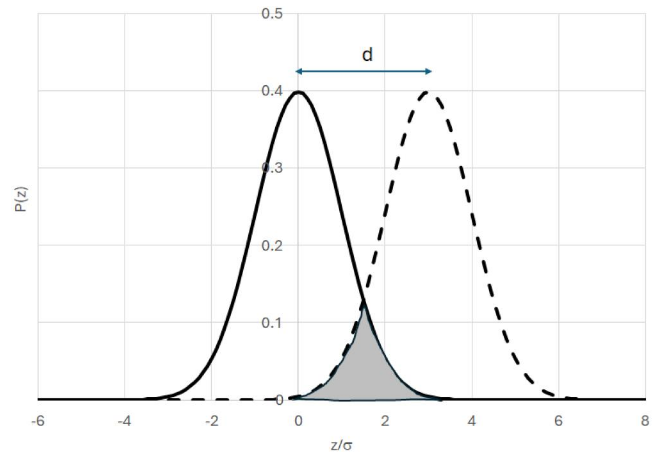

Figure 1. The overlap coefficient, $C(d)$, is shown by the gray shaded area.

$$C(d) = \int_{d/2}^{\infty} P(z)dz + \int_{-\infty}^{-d/2} P(z)dz \quad [5]$$

This equation can be simplified so that the overlap coefficient for identical Gaussian distributions becomes

$$C(d) = \text{erfc}\left(\frac{d}{2\sqrt{2}\sigma}\right) \quad [6]$$

In this equation, $\text{erfc}(x)$ is the complementary error function.

Similarly, the overlap coefficient for the exponential distribution function can be shown to be

$$C(d) = \exp\left(-\frac{d}{2\sigma}\right) \quad [7]$$

For the Rayleigh distribution, the situation is slightly more complicated because the probability distribution functions on the two different surfaces are not identical. The situation is shown in Fig. 2. It can be seen from the figure that the surface roughness is approximately Gaussian below the center line average, but surface heights above the center line average are substantially non-Gaussian, and there is an effective cutoff above which there are no surface asperities; this type of distribution can be used to model a well "run-in" surface, where the highest asperities have been worn off during the "running-in" period. (*21*)

In the case of the Rayleigh distributions shown in Fig. 2, $C(0)$ is not equal to 1, since the distributions are asymmetric, and in fact it can be shown that

$$C(0) = 2\left(1 - \exp\left(-\frac{1}{2}\right)\right) \approx 0.7869 \quad [8]$$

It has also been found that $C(d) = 0$ if $d > 2\sigma$, and for $0 \le d \le 2\sigma$, $C(d)$ is given by the expression

$$C(d) = 2\left(1 - \exp\left(-\frac{1}{2}\left(\frac{d}{2\sigma} - 1\right)^2\right)\right) \quad [9]$$

Therefore, the normalized overlap coefficient (OVC) for the Rayleigh probability distributions is zero if $d > 2\sigma$, and

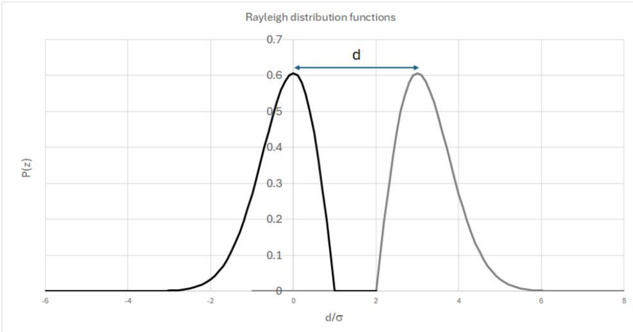

Figure 2. Two rough surfaces, whose centers are separated by a distance d, with Rayleigh probability density functions.

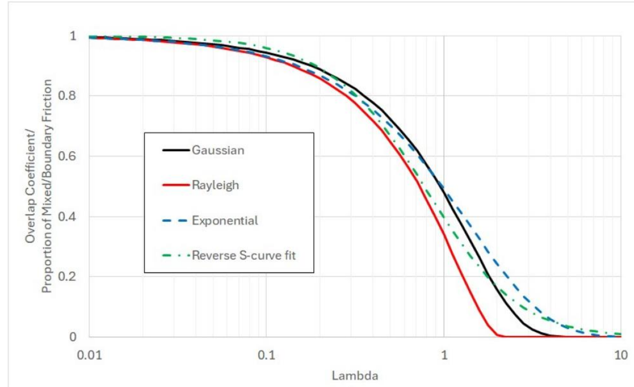

Figure 3. For the Gaussian, exponential, and Rayleigh distributions, the normalized overlap coefficient is plotted against λ. Also shown is the proportion of mixed/boundary friction (X) versus λ predicted by the equation $X = (1 + \lambda^k)^{-a}$, where $k = 3/2$ and $a = 4/3$ (*1*), which is described as "Reverse S-curve fit" on the graph.

for $d < 2\sigma$ is given by

$$OVC = \frac{2\left(1 - \exp\left(-\frac{1}{2}\left(\frac{d}{2\sigma} - 1\right)^2\right)\right)}{2\left(1 - \exp\left(-\frac{1}{2}\right)\right)} \quad [10]$$

The overlap coefficient for two dissimilar Gaussians (i.e., with different variances) is slightly more complex to calculate but has been considered in some detail by Inman and Bradley. (*25*) A fuller discussion of this case can be found in the Appendix.

For a smooth surface contacting a rough surface that has a Gaussian probability density function, the probability density of the smooth surface will be a delta function, $\delta(x)$, and it is known that for a function $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_o)dx = f(x_o) \quad [11]$$

Thus, it can then be shown that the overlap coefficient of a smooth surface contacting a rough surface with a Gaussian probability density function would be simply given by

$$OVC = \exp\left(-\frac{d^2}{2\sigma^2}\right) \quad [12]$$

Linking probability density function overlap coefficients to mixed/boundary friction

In the case of the Gaussian and exponential probability density functions discussed in the previous section, the combined RMS roughness of the two surfaces is $\sqrt{2}\sigma$, so the lambda ratio for these surfaces is simply given by $\lambda = d/(\sqrt{2}\sigma)$. Using the results from the previous section it is found that the normalized overlap coefficient (OVC) for these surfaces is given by the following.

Exponential distribution:

$$OVC = \exp\left(-\frac{\lambda}{\sqrt{2}}\right) \quad [13]$$

Gaussian distribution:

$$OVC = \text{erfc}\left(\frac{\lambda}{2}\right) \quad [14]$$

For the Rayleigh distribution, it turns out that the variance of the probability density function given by Eq. [3] is

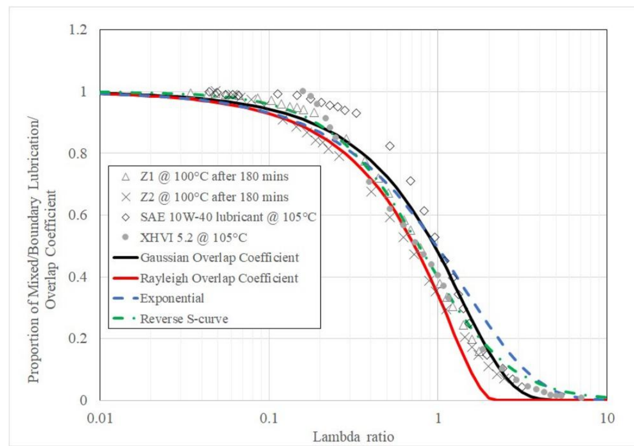

Figure 4. Comparison of experimental normalized mixed/boundary friction versus λ (from mini traction machine experiments reported in Taylor and Sherrington (*1*) and Dawczyk et al. (*19*)) compared to the variation of overlap coefficients with λ, for identical Gaussian distributions and for Rayleigh distributions.

not simply σ, but is rather given by $(2 - \pi/2)\sigma^2$, so in this case, λ is given by the following equation, and the normalized overlap coefficient is given by Eq. [10] in which d/σ can be replaced by $(4 - \pi)\lambda$. It is also observed that there will be no mixed/boundary friction for $d/\sigma > 2$, which implies that there is no mixed/boundary friction for $\lambda > 2.33$.

For Rayleigh surfaces,

$$\lambda = \frac{d}{(4 - \pi)\sigma} \quad [15]$$

It is useful to plot the normalized overlap coefficient versus λ, and compare the variation with that showing the proportion of mixed/boundary friction versus λ (from recent theory (*1*) based on experimental data (*19*)), as shown in Fig. 3.

In Fig. 4, experimental data from references (*1*) and (*19*) have been replotted, along with the normalized overlap coefficient from identical Gaussian distributions (Eq. [14]) and from the Rayleigh distributions of Fig. 2 (Eq. [10]). Note that the experimental data shown in Fig. 4 did not include any lubricants containing friction modifiers.

Figure 4 shows that use of the Rayleigh distribution underestimates the amount of mixed/boundary lubrication in the contact for $\lambda > 1.5$ and so suggests that the surfaces in the experimental data have not fully "run in." It is possible that a more general Weibull distribution could be used for partially "run-in" surfaces, and this will be considered in a future study.

Discussion

Increased pressure on reducing CO_2 emissions is focusing efforts to improve machine efficiency. Examples of improved machine efficiency include the improvement in fuel economy of conventional internal engines along with improving the transmission efficiency of gear systems and pump efficiency and so on in hydraulic systems. This has led to original equipment manufacturers (OEMs) developing downsized machines, filled with lower viscosity lubricants, equipped with smaller oil sumps, and to a move to higher temperature, higher load operation. These efforts will tend to improve machine efficiency, and so reduce frictional losses, but will also tend to increase the proportion of time that machine components spend operating in the mixed and boundary lubrication regimes.

Consequently, there is increasing interest in being able to accurately predict mixed and boundary friction, and a recent review (*26*) summarized some of the many equations that have been used to predict the proportion of mixed/boundary friction in a contact.

In this work, the overlap coefficients of different probability density functions relevant to real engineering rough surfaces have been calculated, and it has been found that these simple functions can also provide a good fit to experimental data (Fig. 4) but without the need for any fitting constants. Note that the experimental data on which the fits are based did not include any "friction-modified" lubricants. Previous work (*2*) has discussed how friction modifiers could potentially be modeled using a modified "reverse S-curve function." Further work is needed to include friction-modified lubricants within the more generalized framework reported here.

Although this article has focused on probability distribution functions that lead to overlap coefficients that can be expressed analytically, there is also the potential to experimentally measure the probability density functions of both new and "run-in" rough surfaces (as reported in Leighton et al. (*22*)), and to then numerically calculate the overlap coefficients of these distributions and compare them to mixed and boundary friction measurements on those surfaces.

A potential weakness of this work is that although it has been assumed that the rough surface asperities interfere with each other as the surfaces approach each other (which could happen as the load on the contact increases), no physical model has been put forward for how the asperities interact with each other (either elastically and/or plastically), although it must be said that there are many other models of this type in the tribological literature.

On the other hand, a strength of this work is that probability density functions for rough surfaces can easily be measured experimentally, and that the approach put forward in this article can be used to simply calculate overlap coefficients of these distributions to predict the variation of mixed/boundary friction as a function of λ. (This can be done either analytically for the case of Gaussian rough surfaces or numerically for experimentally measured rough surfaces.)

The lambda ratio is widely used as a simple tool to assess the lubrication regime of a contact where the film thickness is known by either measurement or prediction. The limits set for each form of contact are effectively arbitrary. However, for surfaces with a Gaussian height distribution, it is realistic to assume the limits of the forms of lubricating films on the basis of the properties of a Gaussian distribution. For example, since 99.7% of the area of a distribution falls within +3 standard deviations of the mean value, if surfaces are separated by at least three times the combined RMS roughness (i.e., $\lambda = 3$), significant asperity contact is unlikely, so a full fluid film is often assumed for such a case However, some tribologists believe full separation of surfaces only occurs for $\lambda > 4$ and some researchers have reported that mixed friction persists for even higher values of λ. (*27*) In general, engineering surfaces that have been well "run in," where the highest asperity peaks have been worn off, will experience significantly lower levels of mixed friction compared to brand new surfaces. However, the precise value of λ at which rough surfaces become fully separated by a fluid film will clearly depend on the character of the surface height distribution and will be lower for smoother surfaces, such as those that have been run in, and higher for others, for example, surfaces produced by some single point cutting processes that have "peaky" height distributions.

In the case of identical Gaussian surfaces, a very simple expression relating the proportion of mixed/boundary friction in a contact, X, to the λ value has been found to be $X = \mathrm{erfc}(\lambda/2)$. Unlike a previously reported fit to experimental data, (*1*) there are no fitting constants in this simple equation, while there is still a good fit to recent experimental data on non-friction-modified lubricants (from mini traction machine experiments (*1*,*19*)). There is a practical advantage for tribologists in having simple equations such as this for being able to predict mixed/boundary friction.

"Run-in" surfaces were also studied, using a Rayleigh probability density function as a model for such surfaces, and it was found that for such surfaces there would be a cutoff value of λ above which mixed/boundary friction would be zero (since there are no more asperities above a certain height for the "run-in" surfaces). In fact, carrying out careful experimental mixed/boundary friction measurements in the range $2 < \lambda < 4$ could be very useful in aiding our understanding of mixed/boundary friction, since although tribology textbooks typically state that surfaces are fully separated when λ is greater than about 3 or 4, other recent published work (*27*) suggests that mixed/boundary friction persists to larger values of λ than this (and this would be expected for a rough surface probability density function with a wider range of surface heights, such as the exponential density function).

The approaches described in this article open the way toward modeling the extremes of friction behavior that might be encountered in newly manufactured components (which could be modeled by the contact of Gaussian surfaces) and the operation of run-in components (which could be modeled by the Rayleigh distribution). The challenge of understanding the impact of "running in" on friction, and the subsequent associated power loss on component operation, is a common one. The methods described in this article provide a simple approach to readily make such an assessment. Friction forces and power losses in machine components can be readily calculated, using a typical load-sharing approach, in which the friction force, F, for mixed/boundary friction would be given by $F = f_o XW$, where X is the proportion of mixed/boundary friction in a contact (as calculated by the methods described here), f_o is the friction coefficient when $\lambda = 0$, and W is the load acting on the contact. The power loss would then be estimated by multiplying the friction force by the sliding speed. By having a more accurate description of how X varies with λ, which is also consistent with experimental data, it is hoped that more reliable friction force and power loss calculations will be possible.

It is also possible that the approach described here may be extended to rough surface wear prediction. In some respects, the research reported here bears some resemblance to the useful recent work of Varenberg (*28*) in which the Archard wear equation was modified to take account of the bearing ratio curves of rough surfaces, and Rabinowicz has also reported on how to relate wear to mixed/boundary friction coefficients. (*29*)

Conclusions

Although rough surface probability density distributions have been used in mixed/boundary friction models since the 1960s, (*11–13*) calculation of the overlap coefficient of such distributions has not previously been reported for tribological applications.

In this study, analytical equations have been derived for the overlap coefficients of Gaussian distributions (for both identical and dissimilar Gaussians), exponential distributions, and the Rayleigh distribution (which has been used as a model probability density distribution for "run-in" surfaces).

The normalized overlap coefficient for rough surfaces with an identical Gaussian distribution has been shown to take an especially simple form, in which $\text{OVC} = \text{erfc}(\lambda/2)$, and it has been shown that this function also gives a good fit to recently published mixed/boundary friction mini traction machine measurements made on a range of different lubricants. (*1*,*19*)

For "run-in" surfaces, which have been modeled assuming the probability density function takes the form of a Rayleigh distribution, it has also been found that there will be no mixed/boundary friction for $\lambda > 2.3$, due to the lack of higher asperities on these idealized "run-in" surfaces.

From a practical perspective, it is possible to experimentally measure rough surface probability density functions (see for example (*22*)) and then to use the experimentally determined distributions to calculate an overlap coefficient numerically.

The main benefit of the work described in this article is that it highlights relatively simple equations that are consistent with experimental data, and that can easily be used by tribologists for the accurate prediction of mixed/boundary friction.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix

The overlap coefficient for dissimilar Gaussians

Henman and Bradley (*25*) have previously studied the overlap coefficient for both identical Gaussian distributions and Gaussian distributions with different means and variances.

Define $P_1(\mu_1, \sigma_1, z)$ and $P_2(\mu_2, \sigma_2, z)$ as

$$P_1(\mu_1, \sigma_1, z) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(z-\mu_1)^2}{2\sigma_1^2}\right) \quad [A.1]$$

$$P_2(\mu_2, \sigma_2, z) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(z-\mu_2)^2}{2\sigma_2^2}\right) \quad [A.2]$$

These two Gaussian distributions intersect at points Z_1, Z_2 given by

$$(Z_1, Z_2) = \frac{\left(\mu_1\sigma_2^2 - \mu_2\sigma_1^2\right) \pm \sigma_1\sigma_2\sqrt{(\mu_1-\mu_2)^2 + (\sigma_2^2-\sigma_1^2)\log_e\left(\frac{\sigma_2^2}{\sigma_1^2}\right)}}{\sigma_2^2 - \sigma_1^2} \quad [A.3]$$

It is worthwhile considering a specific example, where $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = d$, $\sigma_2 = 0.5$, and some examples are shown in Fig. A1 of the overlap of the distributions for different values of d.

The overlap coefficient $C(d)$ can be evaluated relatively straightforwardly, either using the mathematical equation that follows here, or by using Excel's NORMDIST function (which is a useful statistical function for normal distributions), also shown here. In both expressions Z_1 is the smaller root of Eq. [A.3] and Z_2 is the larger root.

$$C(d) = \frac{1}{2}\left(1 + \operatorname{erf}\frac{(Z_1-\mu_2)}{\sigma_2\sqrt{2}}\right) + \frac{1}{2}\left(\operatorname{erf}\frac{(Z_2-\mu_1)}{\sigma_1\sqrt{2}} - \operatorname{erf}\frac{(Z_1-\mu_1)}{\sigma_1\sqrt{2}}\right) + \frac{1}{2}\left(1 - \operatorname{erf}\frac{(Z_2-\mu_2)}{\sigma_2\sqrt{2}}\right) \quad [A.4]$$

$$\begin{aligned} C(d) = {} & \mathrm{NORMDIST}(Z_1, d, \sigma_2, 1) \\ & + (NORMDIST(Z_2, 0, \sigma_1, 1) - NORMDIST(Z_1, 0, \sigma_1, 1)) \\ & + (1 - NORMDIST(Z_2, d, \sigma_2, 1)) \end{aligned} \quad [A.5]$$

where erf(x) is the error function, and for the Excel function NORMDIST(z, μ, σ, FLAG), z is the position at which the cumulative probability value is required, μ is the mean of the normal distribution, σ is the standard deviation, and if FLAG = TRUE (or 1), the cumulative value is calculated. If the FLAG = FALSE (or 0) the value of the normal distribution at z is returned.

It is also worth pointing out that lambda is defined by

$$\lambda = \frac{d}{\sqrt{\sigma_1^2 + \sigma_2^2}} \quad [A.6]$$

When $d = 0$, for the specific example just shown, it is found that the overlap coefficient $C(0)$ is approximately 0.6773. Figure A2 shows the variation in overlap coefficient with λ for different values of σ_1 and σ_2, in comparison with the expected variations for identical Gaussians ($\sigma_1 = \sigma_2$) and also for the case where a Gaussian rough surface contacts a perfectly smooth surface (Eq. [11]).

In addition, Table A1 compares the value of X for different values of λ for dissimilar Gaussians (with $\sigma_1 = 1$ and $\sigma_2 = 0.8$, and also for $\sigma_1 = 1$ and $\sigma_2 = 0.5$) with the values expected for identical Gaussian surfaces and also for a Gaussian surface against a flat surface.

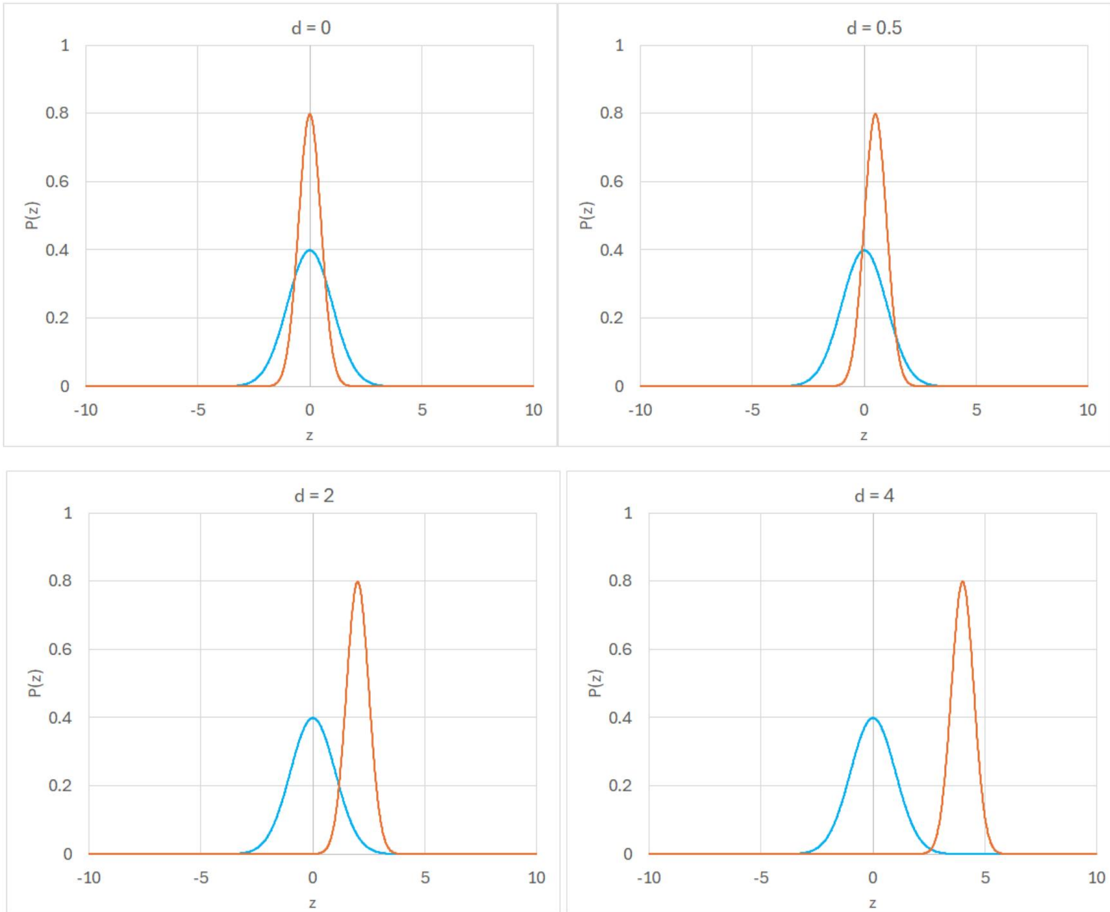

Figure A1. Plot showing overlap of two dissimilar Gaussian distributions, one centered on $z = 0$, with a standard deviation of 1, and the other centered on $z = d$, with a standard deviation of 0.5.

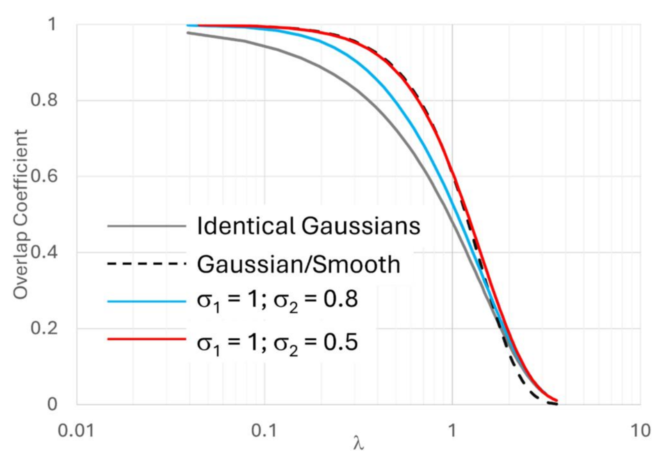

Figure A2. Graph showing the variation of the overlap coefficient (OVC) for identical Gaussian distributions (Eq. [14]), and for a Gaussian distribution contacting a smooth surface (Eq. [11], and also two dissimilar Gaussian distributions, one with $\sigma_1 = 1$ and $\sigma_2 = 0.8$ and the other with $\sigma_1 = 1$ and $\sigma_2 = 0.5$ (these curves have been calculated using the methods described in the preceding).

Table A1. Comparison of simple equations for identical Gaussian rough surfaces, and for a Gaussian surface against a flat surface, compared to numerical calculations of normalized overlap coefficients for dissimilar Gaussian rough surfaces, for selected values of λ.

λ	Identical Gaussians: $X = \mathrm{erfc}\left(\frac{\lambda}{2}\right)$	Gaussian against flat surface: $X = \exp\left(-\frac{\lambda^2}{2}\right)$	Dissimilar Gaussians: $\sigma_1 = 1$, $\sigma_2 = 0.8$	Dissimilar Gaussians: $\sigma_1 = 1$, $\sigma_2 = 0.5$
0.5	0.724	0.883	0.796	0.877
1.0	0.480	0.607	0.529	0.609
1.5	0.289	0.325	0.318	0.361
2.0	0.157	0.135	0.170	0.186
2.5	0.0771	0.0439	0.0834	0.0862
3.0	0.0339	0.0111	0.0361	0.0347