Automorphism groups and the isomorphism problem for frame p-groups of maximal class

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Abstract

The classification of p-groups of maximal class is a deep and long-standing problem in group theory. Recently, a new approach was introduced by Eick, Komma & Saha: this uses the Lazard correspondence to construct the so-called *frame p-groups*. The set of all frame p-groups forms a significant, but still controllable part of the set of all p-groups of maximal class and hence seems a good basis for a classification. This paper investigates the automorphism groups and the isomorphism problem for frame p-groups.

1 Introduction

The classification of p-groups of maximal class is a long-standing project. It was initiated by Blackburn [2], who determined the 2-groups and the 3-groups of maximal class. Alperin [1] proved that for every prime p there is a constant $m(p)$ such that every p-group of maximal class has derived length at most $m(p)$. These two foundations initiated the interest in the p-groups of maximal class and their classification.

The p-groups of maximal class can be visualised via their associated graph $\mathcal{G}(p)$. The vertices of this graph correspond one-to-one to the isomorphism types of p-groups of maximal class and there is an edge $G \to H$ if $H/Z(H) \cong G$ holds. If there exists $H \in \mathcal{G}(p)$ with $H/Z(H) \cong G$, then G is called *capable*. If G is not capable, then it is *terminal*. A group H is a *descendant* of G if there is a path from G to H in the graph.

It is well-known that the graph $\mathcal{G}(p)$ consists of a single isolated point corresponding to the cyclic group C_{p^2} and an infinite tree with root C_p^2. The infinite tree has a unique infinite path $S_2 \to S_3 \to \dots$ starting at its root. Its branch $\mathcal{B}_i$ is the subtree consisting of all descendants of S_i that are not descendants of S_{i+1}. The classification of p-groups of maximal class now translates to understanding the branches $\mathcal{B}_2, \mathcal{B}_3, \dots$. We write $\mathcal{C}_i$ for the full subgraph of $\mathcal{B}_i$ consisting of all capable groups. Then $\mathcal{C}_i$ describes the broad features of $\mathcal{B}_i$ and $\mathcal{B}_i$ contains the fine details.

For $p \geq 5$, the branches in $\mathcal{G}(p)$ have growing depth and for $p \geq 7$ these branches have growing width; these phenomena underline the difficulties inherent in investigating the branches. Eick, Komma & Saha [5] suggested to approach this problem by introducing *frames*: the frame $\mathcal{F}_i$ is a full subtree of $\mathcal{B}_i$ consisting of groups defined from Lie rings; see Section 1.1 for details. Theorems 2 and 12 in [5] yield that

$$\mathcal{C}_i \subseteq \mathcal{F}_i \subseteq \mathcal{B}_i \quad \text{for} \quad i \geq p+2;$$

that is, all capable groups of $\mathcal{B}_i$ are in the frame. Hence, together with $\mathcal{C}_i$, also the frame $\mathcal{F}_i$ is a substantial subtree of $\mathcal{B}_i$.
Different Lie rings can define isomorphic frame groups; this is the isomorphism problem for frame groups. It is the central aim of this paper to address this problem. Additionally, we describe the automorphism group of frame groups. As a preliminary step, we recall the Lie ring construction for frame groups, see [5, 6] for details. As $\mathcal{G}(2)$ and $\mathcal{G}(3)$ are fully determined, we assume $p \geq 5$ throughout the paper.

1.1 Construction of frame groups

Let $\mathbb{Q}_p$ denote the p-adic rational numbers and let $K = \mathbb{Q}_p(\theta)$, where θ is a primitive pth root of unity over $\mathbb{Q}_p$. Let $\mathcal{O}$ be the maximal order of K and $\mathfrak{p}$ its unique maximal ideal. Then $\mathfrak{p}^i$ is the unique ideal of index p^i in $\mathcal{O}$ for $i \in \mathbb{N}_0$. Let $P = \langle\theta\rangle$ be the multiplicative group generated by θ. Note that θ acts by multiplication on $\mathfrak{p}^i$ and diagonally on $\mathfrak{p}^i \wedge \mathfrak{p}^i$. Hence, using Hom_P to describe $\mathbb{Z}_p[P]$-module homomorphism, we define for $i \in \mathbb{N}_0$

$$H_i = \mathrm{Hom}_P(\mathfrak{p}^i \wedge \mathfrak{p}^i, \mathfrak{p}^{2i+1}) \quad \text{and} \quad \hat{H}_i = \{\gamma \in H_i \mid \gamma \text{ surjective}\}.$$

For $\gamma \in \hat{H}_i$ and $x, y, z \in \mathfrak{p}^i$ write $J_\gamma(x, y, z) = \gamma(\gamma(x \wedge y) \wedge z) + \gamma(\gamma(y \wedge z) \wedge x) + \gamma(\gamma(z \wedge x) \wedge y)$. Let $J(\gamma)$ be the ideal of $\mathcal{O}$ generated by all elements $J_\gamma(x, y, z)$; that is,

$$J(\gamma) = \langle J_\gamma(x, y, z) \mid x, y, z \in \mathfrak{p}^i \rangle.$$

Then there exists $\lambda = \lambda(\gamma) \in \mathbb{N}_0$ with $J(\gamma) = \mathfrak{p}^\lambda$ by [6, Theorem 2]. For $m \in \{i, \ldots, \lambda\}$ we define $L_{i,m}(\gamma)$ as the set $\mathfrak{p}^i/\mathfrak{p}^m$ with natural addition and bracket-multiplication

$$[x + \mathfrak{p}^m,\ y + \mathfrak{p}^m]_\gamma = \gamma(x \wedge y) + \mathfrak{p}^m.$$

As $m \leq \lambda$, it follows that this bracket-multiplication defined on $\mathfrak{p}^i/\mathfrak{p}^m$ satisfies the Jacobi-identity and hence $L_{i,m}(\gamma)$ is a Lie ring of p-power order. Note that γ maps surjectively onto $\mathfrak{p}^{2i+1}$. Hence the bracket-multiplication is trivial if and only if $m \leq 2i + 1$. This also implies that $2i + 1 \leq \lambda(\gamma)$. For $i \in \mathbb{N}_0$ we write

$$\mathcal{L}_i = \{L_{i,m}(\gamma) \mid \ \gamma \in \hat{H}_i \ \text{ and } \ 2i + 1 \leq m \leq \lambda(\gamma)\}.$$

Now let $i \geq p$. Then [5, Theorem 1] asserts that each Lie ring $L_{i,m}(\gamma)$ in $\mathcal{L}_i$ is nilpotent of class at most $p-1$. Thus the Larzard correspondence applies and allows us to define a group $G(L_{i,m}(\gamma))$. Multiplication by θ induces an automorphism on $L_{i,m}(\gamma)$ by construction. It thus also induces an automorphism on $G(L_{i,m}(\gamma))$ and, in turn, this allows us to define

$$G_{i,m}(\gamma) = G(L_{i,m}(\gamma)) \rtimes P.$$

The set $\{G_{i,m}(\gamma) \mid \gamma \in \hat{H}_i \text{ and } 2i + 1 \leq m \leq \lambda(\gamma)\}$ is contained in $\mathcal{B}_{i+2}$ and defines a full subgroup of this branch, see Theorem 12 in [5]. This subgraph is the *frame* $\mathcal{F}_{i+2}$. As outlined above, the frame contains all capable groups in $\mathcal{B}_{i+2}$ and hence is a substantial subgraph.

1.2 Isomorphisms and automorphisms

The isomorphism problem for frame groups asks when $G_{i,m}(\gamma) \cong G_{i',m'}(\gamma')$ holds. This will be essential in understanding the frames in the graphs of p-groups of maximal class. The automorphism group of $G_{i,m}(\gamma)$ will be of significance in the determination of central extensions of the frame group and thus, ultimately, for constructing the full branch from its frame.

A special automorphism conjecture

Our results on isomorphisms and automorphisms are all based on a central conjecture. To describe this, let $i \geq p$, let $\gamma \in \hat{H}_i$ and $2i+1 \leq m \leq \lambda(\gamma)$. Consider $G = G_{i,m}(\gamma) = G(L_{i,m}(\gamma)) \rtimes P$ and recall that this is a group in $\mathcal{F}_{i+2}$. The Lazard correspondence allows to identify $G(L_{i,m}(\gamma)) = L_{i,m}(\gamma) = \mathfrak{p}^i/\mathfrak{p}^m$ as sets. Write $\kappa = \theta - 1$ so that that $\mathfrak{p}^j = \kappa^j \mathcal{O}$ for $j \in \mathbb{N}_0$ and define $t \in G$ as the element corresponding to $\kappa^i + \mathfrak{p}^m$. Further, $P = \langle \theta \rangle$ and we define $a \in G$ as the element corresponding to θ. Then $G = \langle a, t \rangle$ follows.

Conjecture 1.1. *Let $i \geq p+2$ and $G \in \mathcal{F}_i$. Then there exists an automorphism $\pi \in \mathrm{Aut}(G)$ with $\pi(a) = at$.*

We have verified Conjecture 1.1 for small p and i. In particular, for $p = 5$ we have verified it for all $i \leq 32$. A proof of Conjecture 1.1 for $p = 5$ is due to Komma [7]. Conjecture 1.1 also holds in the special case of so-called one-parameter skeleton groups by Leedham-Green & McKay [11, Theorem 4.4].

Isomorphisms

We first consider the isomorphism problem. Let $\mathcal{U}$ denote the unit group of the maximal order $\mathcal{O}$ in the field K and let Gal denote the Galois group of K. It is well-known that Gal is cyclic of order $p-1$ and consists of automorphisms mapping $\theta \to \theta^j$ for $1 \leq j \leq p-1$ and $\mathcal{U} = \langle \omega, \theta \rangle (1 + \mathfrak{p}^2)$, where ω is a primitive $(p-1)$st root of unity in $\mathbb{Z}_p$, see [13, Proposition II.5.3]. Each Galois automorphism of K maps $\mathcal{U} \to \mathcal{U}$ and this allows to define the group $\mathcal{U} \rtimes \mathrm{Gal} = \{(u, \sigma) \mid u \in \mathcal{U}, \sigma \in \mathrm{Gal}\}$ with multiplication $(u_1, \sigma_1)(u_2, \sigma_2) = (u_1 \sigma_1^{-1}(u_2), \sigma_1 \sigma_2)$. For $\gamma \in \hat{H}_i$ and $(u, \sigma) \in \mathcal{U} \rtimes \mathrm{Gal}$ we define the map $(u, \sigma) \cdot \gamma = u \cdot (\sigma \cdot \gamma)$ via the following for $x, y \in \mathfrak{p}^i$

$$\begin{aligned} u \cdot \gamma(x \wedge y) &= u\gamma((u^{-1}x) \wedge (u^{-1}y)), \\ \sigma \cdot \gamma(x \wedge y) &= \sigma(\gamma(\sigma^{-1}(x) \wedge \sigma^{-1}(y))). \end{aligned}$$

As multiplication by $\mathcal{U}$ and the action by Gal leave each ideal $\mathfrak{p}^i$ setwise invariant, it follows that this induces an action of $\mathcal{U} \rtimes \mathrm{Gal}$ on $\hat{H}_i$. For two elements $\gamma, \delta \in \hat{H}_i$ we write $\gamma \equiv \delta \bmod \mathfrak{p}^m$ if $\gamma(x \wedge y) \equiv \delta(x \wedge y) \bmod \mathfrak{p}^m$ for all $x, y \in \mathfrak{p}^i$. The following is proved in Section 4.

Theorem 1.2. (Isomorphisms)
Let $i \geq p$ and assume that Conjecture 1.1 holds. Then $G_{i,m}(\gamma) \cong G_{i',m'}(\gamma')$ if and only if $i = i'$, $m = m'$ and there exists $(u, \sigma) \in \mathcal{U} \rtimes \mathrm{Gal}$ with $\gamma' \equiv (u, \sigma) \cdot \gamma \bmod \mathfrak{p}^m$.

Automorphisms

We now consider the automorphism groups of frame groups. As before, let $G = G_{i,m}(\gamma)$ and write $S = \mathrm{Stab}_{i,m}(\gamma) = \{(u,\sigma) \in \mathcal{U} \rtimes \mathrm{Gal} \mid (u,\sigma)\cdot\gamma \equiv \gamma \bmod \mathfrak{p}^m\}$. Lemma 2.2 shows how S induces a subgroup A_S of $\mathrm{Aut}(G)$. Let $S_U = S \cap \mathcal{U}$, where $\mathcal{U}$ is identified with $\{(u,id) \mid u \in \mathcal{U}\}$, and let A_U be the subgroup of $\mathrm{Aut}(G)$ induced by S_U. Recall that G is generated by a, t and write $T = G(L_{i,m}(\gamma)) \leq G$ to shorten notation. Lemma 3.2 shows that T is a characteristic subgroup of G. Define $A_1 = C_{\mathrm{Aut}(G)}(aT) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(a) \equiv a \bmod T\}$ and $A_2 = C_{\mathrm{Aut}(G)}(a) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(a) = a\}$. Then

$$\mathrm{Aut}(G) \geq A_1 \geq A_2\,\mathrm{Inn}(G) \geq A_2 \geq A_2 \cap \mathrm{Inn}(G) \geq \{id\}.$$

The following is proved in Section 3.

Theorem 1.3. (Automorphisms)
Let $i \geq p$ and assume that Conjecture 1.1 holds. Let $G = G_{i,m}(\gamma)$ and assume that G is not on the mainline. Write π for the automorphism in $\mathrm{Aut}(G)$ of Conjecture 1.1.

(a) *A_1 is the unique Sylow p-subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)/A_1 \cong S/S_U$.*
(b) *$A_2\,\mathrm{Inn}(G)$ is normal of index p in A_1 and $A_1 = \langle\pi\rangle A_2\,\mathrm{Inn}(G)$.*
(c) *$A_2 = A_U$.*
(d) *$A_2 \cap \mathrm{Inn}(G)$ has order p and is generated by the conjugation with a.*

Remark 1.4. If G is a mainline group, then parts (b)-(d) still hold, but part (a) does not. If Conjecture 1.1 does not hold, then $A_1 = A_2\,\mathrm{Inn}(G)$ follows.

2 The action of $\mathcal{U} \rtimes \mathrm{Gal}$

The aim of this section is to investigate the action of $\mathcal{U} \rtimes \mathrm{Gal}$ on $\hat{H}_i$. Recall that $\lambda(\gamma)$ is defined by $J(\gamma) = \mathfrak{p}^{\lambda(\gamma)}$ for $\gamma \in \hat{H}_i$.

Lemma 2.1. *Let $u \in \mathcal{U}$, let $\sigma \in \mathrm{Gal}$ and let $\gamma \in \hat{H}_i$.*

(a) *$(u,\sigma)\cdot\gamma$ is an element in $\hat{H}_i$.*
(b) *$\lambda((u,\sigma)\cdot\gamma) = \lambda(\gamma)$.*
(c) *The map $L_{i,m}(\gamma)) \mapsto L_{i,m}((u,\sigma)\cdot\gamma)$ defines an action of $\mathcal{U} \rtimes \mathrm{Gal}$ on $\mathcal{L}_i$.*

Proof. (a) First consider the action by $\mathcal{U}$. If $u \in \mathcal{U}$ and $x, y \in \mathfrak{p}^i$, then $u^{-1}x \in \mathfrak{p}^i$. Hence $u\cdot\gamma : \mathfrak{p}^i \wedge \mathfrak{p}^i \to \mathfrak{p}^{2i+1}$ and this is a surjective homomorphism. Further, if γ is compatible with P, then $u\cdot\gamma$ is also. Hence $u\cdot\gamma \in \hat{H}_i$. A similar argument shows that $\sigma\cdot\gamma \in \hat{H}_i$.
(b) Recall that $J(\gamma)$ is the ideal generated by the elements $J_\gamma(x,y,z) = \gamma(\gamma(x \wedge y) \wedge z) + \gamma(\gamma(y \wedge z) \wedge x) + \gamma(\gamma(z \wedge x) \wedge y)$ for $x, y, z \in \mathfrak{p}^i$. Let $u \in \mathcal{U}$ and write $\delta = u\cdot\gamma$. Then $\delta(ux \wedge uy) = u\,\gamma(x \wedge y)$. Applying this twice, we obtain that $\delta(\delta(ux \wedge uy) \wedge uz) = \delta(u\gamma(x \wedge y) \wedge uz) = u\,\gamma(\gamma(x \wedge y) \wedge z)$. Thus $J_\delta(ux, uy, uz) = uJ_\gamma(x,y,z)$. As x, y, z are arbitrary elements in $\mathfrak{p}^i$, it follows that $J(\delta) = J(\gamma)$. Thus $\lambda(\delta) = \lambda(\gamma)$ as desired. A similar argument shows that $\lambda(\sigma\cdot\gamma) = \lambda(\gamma)$.
(c) follows directly from (a) and (b). □

Our next aim is to describe the isomorphisms of frame groups induced by the action of $\mathcal{U} \rtimes \mathrm{Gal}$ on $\hat{H}_i$. Recall that the frame group $G_{i,m}(\gamma)$ is generated by a, t, where a corresponds to multiplication by θ and $t \in G(L_{i,m}(\gamma))$ corresponds to $\kappa^i + \mathfrak{p}^m$. The group $G(L_{i,m}(\gamma))$ can be identified as set with $L_{i,m}(\gamma) = \mathfrak{p}^i/\mathfrak{p}^m$.

Lemma 2.2. *Let $\gamma \in \hat{H}_i$, let $u \in \mathcal{U}$ and let $\sigma \in \mathrm{Gal}$ with $\sigma(\theta) = \theta^r$.*

(a) *The map $\nu_u : \mathfrak{p}^i/\mathfrak{p}^m \to \mathfrak{p}^i/\mathfrak{p}^m : x + \mathfrak{p}^m \mapsto ux + \mathfrak{p}^m$ induces an isomorphism $G_{i,m}(\gamma) \to G_{i,m}(u \cdot \gamma)$ mapping $a \mapsto a$ and $t \mapsto ut$.*

(b) *The map $\mu_\sigma : \mathfrak{p}^i/\mathfrak{p}^m \to \mathfrak{p}^i/\mathfrak{p}^m : x + \mathfrak{p}^m \mapsto \sigma(x) + \mathfrak{p}^m$ induces an isomorphism $G_{i,m}(\gamma) \to G_{i,m}(\sigma \cdot \gamma)$ mapping $a \mapsto a^r$ and $t \mapsto \sigma(t)$.*

Proof. (a) Since $u \in \mathcal{U}$, multiplication by u leaves $\mathfrak{p}^i$ and $\mathfrak{p}^m$ invariant, and induces a Lie ring isomorphism $L_{i,m}(\gamma) \to L_{i,m}(u \cdot \gamma)$. The Lazard correspondence translates this to a group isomorphism $G(L_{i,m}(\gamma)) \to G(L_{i,m}(u \cdot \gamma))$. The multiplication with u satisfies $u\theta = \theta u$. Hence we obtain a group isomorphism $G_{i,m}(\gamma) \to G_{i,m}(u \cdot \gamma)$ with $a \mapsto a$ and $t \mapsto ut$.
(b) This follows similarly to (a). Note that $\sigma(\theta) = \theta^r$ and thus μ_σ induces an isomorphism mapping $a \mapsto a^r$. □

Lemma 2.2 has the following direct consequence.

Corollary 2.3. *Let $\gamma \in \hat{H}_i$, let $u \in \mathcal{U}$ and let $\sigma \in \mathrm{Gal}$ with $\sigma(\theta) = \theta^r$ so that $(u, \sigma) \in Stab_{i,m}(\gamma)$. Then there exists an automorphism $\iota : G_{i,m}(\gamma) \to G_{i,m}(\gamma)$ with $\iota(a) = a^r$ and $\iota(t) = \sigma(ut)$.*

Proof. If $(u, \sigma) \in Stab_{i,m}(\gamma)$, then $(u, \sigma) \cdot \gamma \equiv \gamma \bmod \mathfrak{p}^m$ and $G_{i,m}(\gamma) = G_{i,m}((u, \sigma) \cdot \gamma)$. The result now follows by Lemma 2.2. □

3 Automorphisms of frame groups

The aim of this section is to describe the automorphism group of a frame group and to prove Theorem 1.3. Throughout, we assume that $i \geq p$ and that Conjecture 1.1 holds. We consider $\gamma \in \hat{H}_i$ and write $G = G_{i,m}(\gamma)$ and $T = G(L_{i,m}(\gamma))$ to shorten notation.

Lemma 3.1. *If $G_{i,m}(\gamma) \cong G_{i',m'}(\gamma')$, then $i = i'$ and $m = m'$ hold.*

Proof. Suppose that $G_{i,m}(\gamma) \cong G_{i',m'}(\gamma')$. By Theorem 12 in [5], $G_{i,m}(\gamma)$ is a group in $\mathcal{B}_{i+2}$. This yields $i = i'$. By construction, $|G_{i,m}(\gamma)| = p^{m-i+1}$ which, in turn, implies $m = m'$. □

Lemma 3.2. *$T = C_G(\gamma_2(G)/\gamma_4(G))$ and hence T is fully invariant.*

Proof. First note that $G' = \gamma_2(G) = \mathfrak{p}^{i+1}/\mathfrak{p}^m$ and $\gamma_4(G) = \mathfrak{p}^{i+3}/\mathfrak{p}^m$ as sets. Since $2i + 1 \geq i + 3$, it follows that $T/\gamma_4(G)$ is abelian and hence T centralizes $\gamma_2(G)/\gamma_4(G)$. Each 2-step centraliser is a maximal subgroup in a p-group of maximal class and hence $T = C_G(\gamma_2(G)/\gamma_4(G))$. This implies directly that T is fully invariant. □

Lemma 3.3. *$G' = \{[g, a] \mid g \in T\}$.*

Proof. The group G is in $\mathcal{B}_{i+2}$ and thus has order at least $p^{i+2} \geq p^{p+2}$. Thus Theorem 3.3.5 of [12] yields that G has positive degree of commutativity. Lemma 3.3.7(c) of [12] now implies the result. (In the notation of [12] $P_2 = G'$ and $P_1 = T$.) □

Lemma 3.4. *For each $s \in T$ there exists $\alpha_s \in \mathrm{Aut}(G)$ mapping $a \mapsto as$.*

Proof. Recall that $t \in T \setminus G'$. As $[T : G'] = p$, it follows that $T = \{t^x w \mid 0 \leq x < p, w \in G'\}$. Hence $s = t^x w$ with $w \in G'$ and $0 \leq x < p$. Recall that $\pi(a) = at$, where π is defined in Conjecture 1.1. Thus $\pi^x(a) = at^x r_x$ for $0 \leq x < p$ and some $r_x \in G'$. Lemma 3.3 yields that for $v \in G'$ there exists $g_v \in T$ with $v^{-1} = [g_v, a]$. Let ι_g denote conjugation by g and note that $\iota_{g_v}(a) = a^{g_v} = av$. Thus $\pi^x(\iota_{g_v}(a)) = \pi^x(av) = at^x r_x \pi^x(v)$. Hence choosing $v = \pi^{-x}(r_x^{-1} w)$ yields that $\alpha_s = \pi^x \iota_{g_v}$ maps a to as. □

Lemma 3.5. *$\alpha \in \mathrm{Aut}(G)$ satisfies $\alpha(a) = a$ if and only if there exists $(u, id) \in S_U \leq \mathrm{Stab}_{i,m}(\gamma)$ inducing α.*

Proof. If $(u, id) \in \mathrm{Stab}_{i,m}(\gamma)$, then this induces an automorphism of G with $\alpha(a) = a$ by Corollary 2.3. It remains to show the converse.
Assume that $\alpha \in \mathrm{Aut}(G)$ with $\alpha(a) = a$. Then α induces $\beta \in \mathrm{Aut}(T)$ by restriction, since T is characteristic. Recall that $T = G(L_{i,m}(\gamma))$. The Lazard correspondence translates β to an automorphism (which we also call β) of the Lie ring $L_{i,m}(\gamma)$. This automorphism has the form $\beta : \mathfrak{p}^i/\mathfrak{p}^m \to \mathfrak{p}^i/\mathfrak{p}^m : x + \mathfrak{p}^m \mapsto \psi(x) + \mathfrak{p}^m$ for some set map $\psi : \mathfrak{p}^i \to \mathfrak{p}^i$. The Lie-automorphism β is $\mathbb{Z}_p$-linear and compatible with the Lie multiplication. In particular, $\beta([x + \mathfrak{p}^m, y + \mathfrak{p}^m]) = \beta(\gamma(x \wedge y) + \mathfrak{p}^m) = \psi(\gamma(x \wedge y)) + \mathfrak{p}^m$ and $\beta([x + \mathfrak{p}^m, y + \mathfrak{p}^m]) = [\psi(x) + \mathfrak{p}^m, \psi(y) + \mathfrak{p}^m] = \gamma(\psi(x) \wedge \psi(y)) + \mathfrak{p}^m$. Hence, it follows for $x, y \in \mathfrak{p}^i$ and $a, b \in \mathbb{Z}_p$

$$\begin{aligned}
&(1) \quad \psi(\gamma(x \wedge y)) \equiv \gamma(\psi(x) \wedge \psi(y)) \bmod \mathfrak{p}^m,\\
&(2) \quad \psi(ax + by) \equiv a\psi(x) + b\psi(y) \bmod \mathfrak{p}^m,\\
&(3) \quad \psi(\theta x) \equiv \theta\psi(x) \bmod \mathfrak{p}^m,
\end{aligned}$$

where equivalence (3) is based on the observation that $\alpha(ag) = a\alpha(g)$ for $g \in G$ induces $\beta(\theta x) = \theta\beta(x)$. Write $\kappa = \theta - 1$. Recall that $\mathfrak{p}^i = \kappa^i \mathcal{O}$ and an element of $\mathcal{O}$ has the form $x_1 + x_2\theta + \ldots + x_p\theta^{p-1}$ with $x_i \in \mathbb{Z}_p$. Define $w \in \mathcal{O}$ via $\psi(\kappa^i) = w\kappa^i$. The bijectivity of β implies that $w \in \mathcal{U}$. Further,

$$\psi(\kappa^i(x_1 + \ldots + x_p\theta^{p-1})) = w\kappa^i(x_1 + \ldots + x_p\theta^{p-1})$$

and thus the image of κ^i fully defines the map ψ. In summary, β is induced by multiplication with $w \in \mathcal{U}$ and equivalence (1) implies that $w \in \mathrm{Stab}_{i,m}(\gamma)$. □

Lemma 3.6. *If $\alpha \in \mathrm{Aut}(G)$ satisfies $\alpha(a) \equiv a^r \bmod T$, then there exists $(u, \sigma_r) \in \mathrm{Stab}_{i,m}(\gamma)$, where $\sigma_r \in \mathrm{Gal}$ maps $\theta \to \theta^r$.*

Proof. The automorphism α induces $\beta \in \mathrm{Aut}(T)$ by restriction. The Lazard correspondence then translates β to an automorphism (which we also call β) of the Lie ring $L_{i,m}(\gamma)$ which is of the form $\beta : \mathfrak{p}^i/\mathfrak{p}^m \to \mathfrak{p}^i/\mathfrak{p}^m : x + \mathfrak{p}^m \mapsto \psi(x) + \mathfrak{p}^m$ for some map ψ. A similar argument as in Lemma 3.5 shows $\psi(\gamma(x \wedge y)) \equiv \gamma(\psi(x) \wedge \psi(y)) \bmod \mathfrak{p}^m$ and $\psi(ax + by) \equiv a\psi(x) + b\psi(y) \bmod \mathfrak{p}^m$ for $x, y \in \mathfrak{p}^i$ and $a, b \in \mathbb{Z}_p$. Since conjugation by a on

T corresponds to multiplication by θ on $L_{i,m}(\gamma)$, we have that $x^a = \theta x$ holds for $x \in T$. Hence applying α gives $\alpha(x^a) = \alpha(x)^{\alpha(a)} = \alpha(x)^{a^r}$. Therefore,

$$\psi(\theta x) \equiv \theta^r \psi(x) \pmod{\mathfrak{p}^m}.$$

Since $\sigma_r(\mathfrak{p}^i) = \mathfrak{p}^i$, a similar argument as in Lemma 3.5 shows that there exists $u \in \mathcal{U}$ with $\psi(\kappa^i) \equiv u\,\sigma_r(\kappa^i) \pmod{\mathfrak{p}^m}$. This extends similarly to Lemma 3.5 to

$$\psi(x) \equiv u\sigma_r(x) \pmod{\mathfrak{p}^m} \qquad \text{for all } x \in \mathfrak{p}^i.$$

Hence $(u, \sigma_r) \in \mathrm{Stab}_{i,m}(\gamma)$. □

We now prove Theorem 1.3.

Proof **of Theorem 1.3**
Recall that $A_1 = C_{\mathrm{Aut}(G)}(aT)$ and $A_2 = C_{\mathrm{Aut}(G)}(a)$. Note that G is not on the mainline. As $\gamma \in \hat{H}_i$, it follows that $\gamma(\mathfrak{p}^i \wedge \mathfrak{p}^i) = \mathfrak{p}^{2i+1}$ and hence G is not on the main line if and only if $m > 2i+1$. In turn, if $m > 2i+1$, then γ is non-trivial in $\mathfrak{p}^i/\mathfrak{p}^m$. Further, recall that $\mathcal{U} = \langle \omega, \theta \rangle (1 + \mathfrak{p}^2)$, where ω is a $(p-1)$st root of unity in $\mathbb{Z}_p$.

Step (1).
By Lemma 3.5, an automorphism $\alpha \in \mathrm{Aut}(G)$ satisfies $\alpha(a) = a$ if and only if it is induced by some element $(u, id) \in \mathrm{Stab}_{i,m}(\gamma)$. Hence $A_2 = C_{\mathrm{Aut}(G)}(a) = A_U$ which proves (c). Further, as ω acts regularly via $\omega \cdot \gamma(x \wedge y) = \omega\gamma(\omega^{-1}x \wedge \omega^{-1}y) = \omega^{-1}\gamma(x \wedge y)$, it follows that ω and its powers are never contained in $\mathrm{Stab}_{i,m}(\gamma)$ for $m > 2i+1$. Thus $A_2 = A_U$ is a p-group and hence $A_2 \,\mathrm{Inn}(G)$ is a p-group.

Step (2).
Let $\alpha \in A_2 \cap \mathrm{Inn}(G)$. Then α is induced by conjugation with $g \in G$, say. Write $g = a^x s$ for $0 \le x < p$ and $s \in T$. Since $\alpha \in A_2$, it follows that $a^g = a^s = a$. Hence s commutes with a; this implies $s \in Z(G)$. Hence s acts trivially by conjugation and we can assume that $g = a^x$. In summary, α is induced by conjugation with a^x for some $0 \le x < p$. This proves (d).

Step (3).
Clearly, $A_2 \,\mathrm{Inn}(G) \le A_1$. To prove (b), it remains to determine the associated index. We consider the orbit of a for the action of A_1 and the action of $A_2 \,\mathrm{Inn}(G)$; note that A_2 is the stabiliser of a in both cases. First, the orbit of a under A_1 has the form $\{as \mid s \in T\}$ by Lemma 3.4. Hence $|A_1| = |T||A_2|$. Next, the orbit of a under $A_2 \,\mathrm{Inn}(G)$ has the form $\{au \mid u \in G'\}$ by Lemma 3.3. Hence $|A_2 \,\mathrm{Inn}(G)| = |G'||A_2|$. Thus $[A_1 : A_2 \,\mathrm{Inn}(G)] = |T|/|G'| = p$. This proves (b).

Step (4).
By construction, A_1 is the kernel of the action of $\mathrm{Aut}(G)$ on G/T. Thus A_1 is normal in $\mathrm{Aut}(G)$. Since G/T has order p, it follows that $\mathrm{Aut}(G)/A_1$ has order dividing $(p-1)$. Moreover, Lemmas 2.2 and 3.6 imply that $\mathrm{Aut}(G)/A_1 \cong S/S_U$ embeds into Gal. Using (b), we observe that $|A_1| = p|A_2 \,\mathrm{Inn}(G)|$. As A_2 is a p-group by Step (1), it follows that A_1 is a p-group. Sylow's theorems imply that a normal Sylow p-subgroup is unique. In summary, this yields (a). □

Figure 1 visualises the subgroups of $\mathrm{Aut}(G)$.

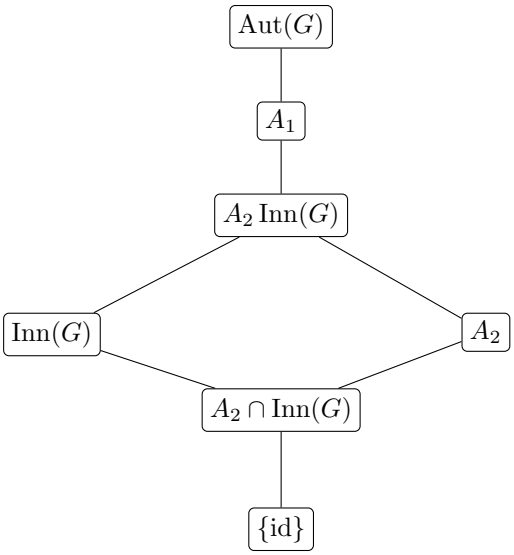


Figure 1: Structure of $\mathrm{Aut}(H)$

4 The isomorphism problem for frame groups

The aim of this section is to investigate the isomorphism problem for frame groups and to prove Theorem 1.2. Throughout, we assume that $i \geq p$ and Conjecture 1.1 holds. We consider two frame groups $G_{i,m}(\gamma)$ and $G_{i',m'}(\gamma')$. If $i = i'$ and $m = m'$ and there exists (u, σ) with $(u, \sigma) \cdot \gamma \equiv \gamma' \bmod \mathfrak{p}^m$, then the two groups are isomorphic by Lemma 2.2. It remains to prove the converse.

Thus we assume now that $G_{i,m}(\gamma) \cong G_{i',m'}(\gamma')$ via an isomorphism ι. Lemma 3.1 yields that $i = i'$ and $m = m'$. We show that there exists $(u, \sigma) \in \mathcal{U} \rtimes \mathrm{Gal}$ mapping γ to γ' modulo $\mathfrak{p}^m$. We write a, t for the generators of the groups in both cases, and let $T_1 = G(L_{i,m}(\gamma))$ and $T_2 = G(L_{i,m}(\gamma'))$. Now $\iota : G_{i,m}(\gamma) \to G_{i,m}(\gamma')$. The image of a under ι is an arbitrary element in $G_{i,m}(\gamma')$. This element can be written as $a^r s$ with $0 \leq r < p$ and $s \in T_2$; thus $\iota(a) = a^r s$.

Step (1).
We apply Lemma 2.2 with the Galois automorphism σ mapping $\theta \to \theta^{-r}$. We obtain $\gamma'' = \sigma \cdot \gamma'$ and $G_{i,m}(\gamma) \cong G_{i,m}(\gamma'')$ via an isomorphism ι' that maps a to ax for some $x \in T_3$, where $T_3 = G(L_{i,m}(\gamma''))$.

Step (2).
We apply Lemma 3.4 and modifiy ι' by an automorphism of $G_{i,m}(\gamma'')$ so that we obtain a new isomorphism $\iota'' : G_{i,m}(\gamma) \to G_{i,m}(\gamma'')$ and this maps a to a.

Step (3).
We now have an isomorphism $\iota'' : G_{i,m}(\gamma) \to G_{i,m}(\gamma'')$ mapping a to a. Similar to Lemma 3.5, there now exists $u \in \mathcal{U}$ so that $u \cdot \gamma \equiv \gamma'' \bmod \mathfrak{p}^m$ inducing ι''.

5 Analysing the automorphism group

Let $i \geq p$ and $\gamma \in \hat{H}_i$. Let $G = G_{i,m}(\gamma)$ not on the mainline. Recall that A_S is the subgroup of $\mathrm{Aut}(G)$ induced by $S = \mathrm{Stab}_{i,m}(\gamma) \leq \mathcal{U} \rtimes \mathrm{Gal}$ and A_U is its subgroup induced by $\mathcal{U}$. Setting $r = [S : S_U]$, Theorem 1.3 yields that

$$\begin{aligned} |\operatorname{Aut}(G)| &= [S : S_U]|A_1| \\ &= r \cdot p \cdot |A_2 \operatorname{Inn}(G)| \\ &= r \cdot p \cdot \frac{|A_2| \cdot |\operatorname{Inn}(G)|}{p} \\ &= r \cdot |A_U| \cdot |G|/p. \end{aligned}$$

A more detailed investigation of $\mathrm{Aut}(G)$ mainly depends on an investigation of A_U. Recall that $\mathcal{U} = \langle \omega, \theta \rangle (1 + \mathfrak{p}^2)$ and $\gamma(\theta x \wedge \theta y) = \theta\gamma(x \wedge y)$, since $\gamma \in \hat{H}_i$ is compatible with θ. Thus for all $x, y \in \mathfrak{p}^i$, it follows that

$$(\omega \cdot \gamma)(x \wedge y) = \omega^{-1}\gamma(x \wedge y) \quad \text{and} \quad (\theta \cdot \gamma)(x \wedge y) = \gamma(x \wedge y).$$

Hence θ always stabilises; it yields the inner automorphism induced by conjugation with a. And ω never stabilises, as G is not on the mainline and thus γ is non-trivial mod $\mathfrak{p}^m$. It remains to consider the subgroup $1 + \mathfrak{p}^2$ of $\mathcal{U}$. Let σ_j be the Galois automorphism defined by $\theta \mapsto \theta^j$ and write $l = (p-1)/2$. For $2 \leq k \leq l$ define

$$\vartheta_k : \mathfrak{p}^i \wedge \mathfrak{p}^i \to \mathfrak{p}^{2i+1} : x \wedge y \mapsto \sigma_k(x)\sigma_{1-k}(y) - \sigma_{1-k}(x)\sigma_k(y).$$

Then [5, Lemma 6] yields $\vartheta_k \in \hat{H}_i$. Hence by [4, Sec 8.1.2], for each $\gamma \in \hat{H}_i$ there exist unique parameters $c_2, \ldots, c_l \in K$

$$\gamma = \sum_{k=2}^{l} c_k \vartheta_k.$$

Define $\rho_k : \mathcal{U} \to \mathcal{U} : u \mapsto u\sigma_k(u)^{-1}\sigma_{1-k}(u)^{-1}$. A straighforward calculation yields for $u \in \mathcal{U}$ and $\sigma \in \mathrm{Gal}$

$$(u, \sigma) \cdot \sum_{k=2}^{l} c_k \vartheta_k = \sum_{k=2}^{l} \sigma(c_k)\rho_k(u)\vartheta_k.$$

We now restrict to *one-parameter* homomorphisms γ; that is, $\gamma = c_k\vartheta_k$ for some $k \in \{2, \ldots, l\}$. As γ and ϑ_k both are surjective onto $\mathfrak{p}^{2i+1}$ by the definition of $\hat{H}_i$, it follows that $c_k \in \mathcal{U}$. Hence

$$(u, id) \in \mathrm{Stab}_{i,m}(\gamma) \quad \Leftrightarrow \quad u \cdot \gamma \equiv \gamma \bmod \mathfrak{p}^m \quad \Leftrightarrow \quad \rho_k(u) \equiv 1 \bmod \mathfrak{p}^m.$$

Hence we have now translated the computation of A_U to the investigation of the number-theoretic maps ρ_k. We refer to [4, Sec 3.4.2] for further details.

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